The Basics of Physics with Calculus – Part II

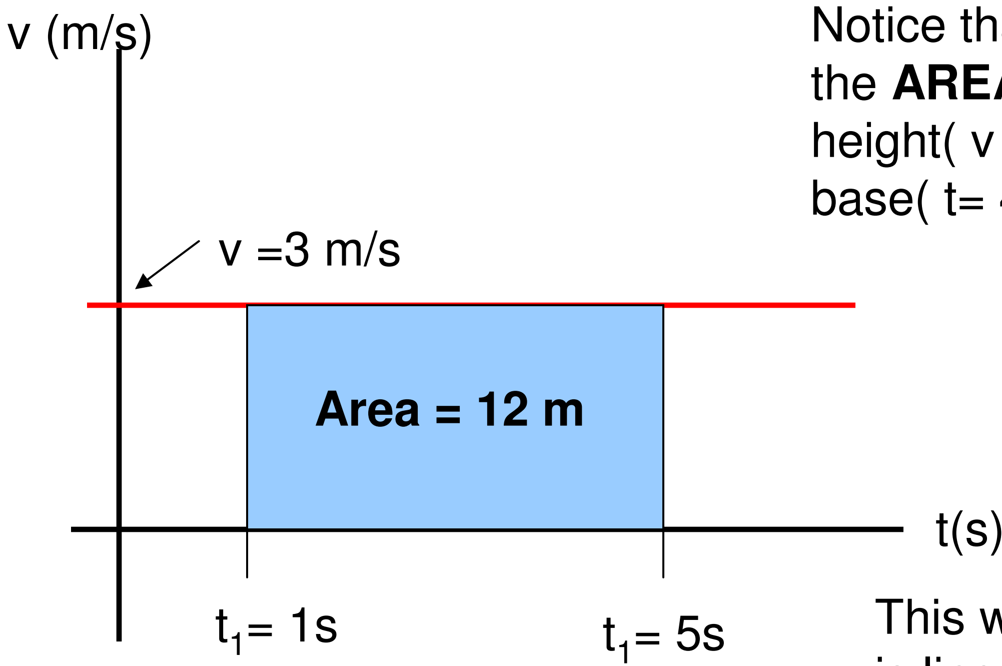
AP Physics C

The "AREA"

We have learned that the rate of change of displacement is defined as the VELOCITY of an object. Consider the graph below

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$v = \frac{dx}{dt}$$

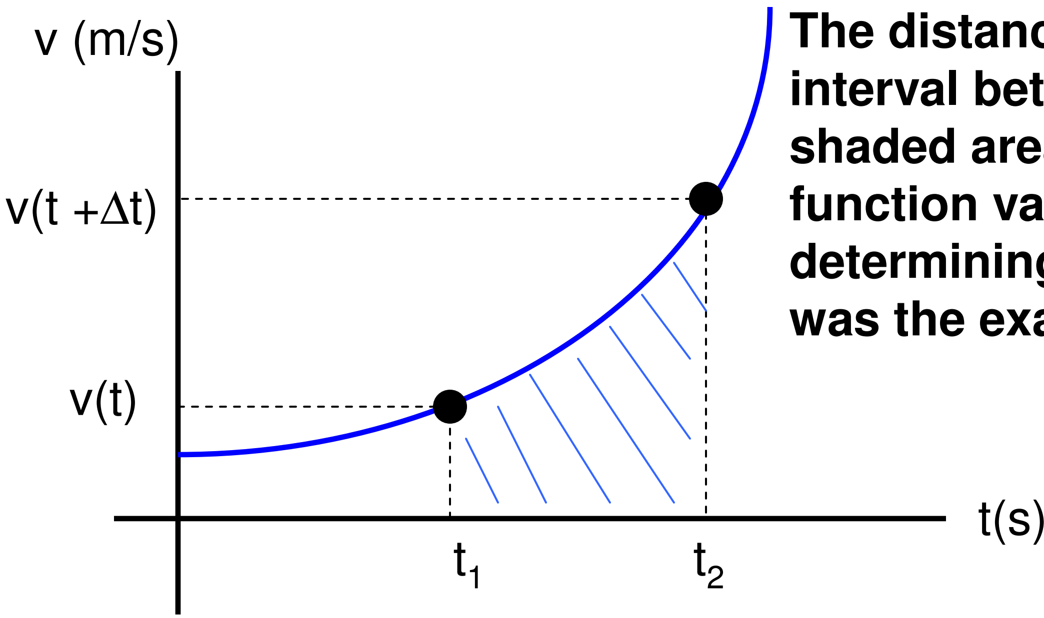


Notice that the 12 m happens to be the **AREA** under the line or the height(v = 3 m/s) times the base(t= 4 seconds) = 12 meters

This works really nice if the function is linear. What if it isn't?

The "Area"

How do we determine HOW FAR something travels when the function is a curve? Consider the velocity versus time graph below

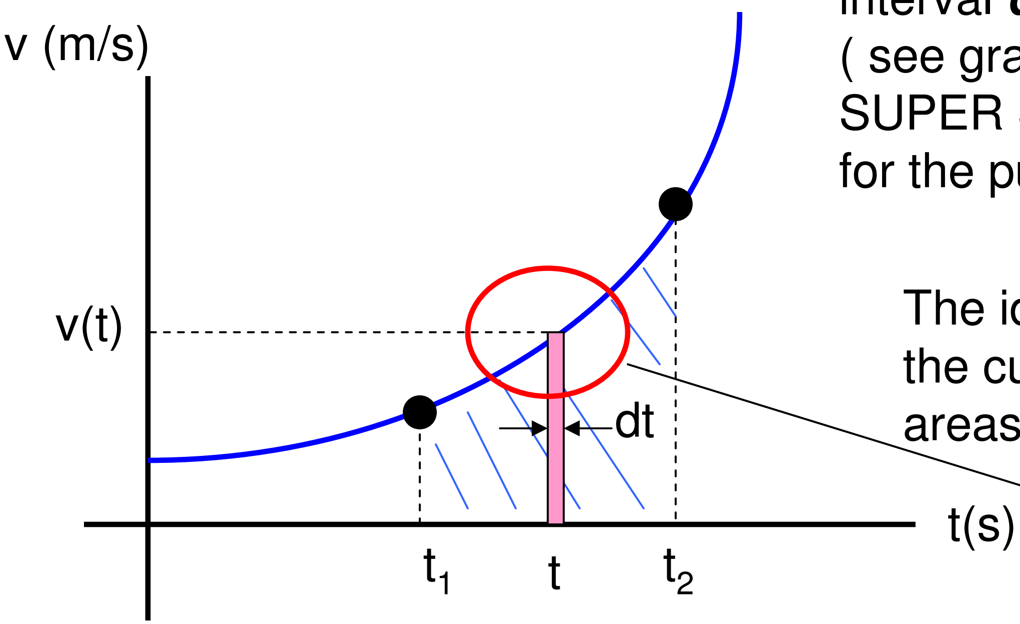


The distance traveled during the time interval between t_1 and t_2 equals the shaded area under the curve. As the function varies continuously, determining this area is NOT easy as was the example before.

So how do we find the area?

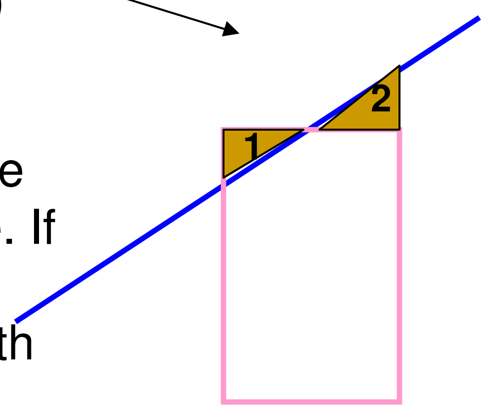
Once again, we ZOOM in…

Consider an arbitrary time ***t***



Place a differential time interval ***dt*** about time ***t*** (see graph). This rectangle is SUPER SMALL and is only visible for the purpose of an explanation.

The idea is that the AREA under the curve is the SUM of all the areas of each individual "***dt***".



With "dt" very small, area 1 fits into area 2 so that the approximate area is simply the area of the rectangle. If we find this area for ALL the small ***dt's*** between ***t1*** and ***t2***, then added them all up, we would end up with the TOTAL AREA or TOTAL DISPLACEMENT.

The "Integral"

The temptation is to use the conventional summation sign "Σ" . The problem is that you can only use the summation sign to denote the summing of **DISCRETE QUANTITIES** and **NOT** for something that is continuously varying. Thus, we cannot use it.

When a continuous function is summed, a different sign is used. It is called and **Integral**, and the symbol looks like this:

$$\int$$

When you are dealing with a situation where you have to integrate realize:

- **WE ARE GIVEN:** the derivative already
- **WE WANT:** The original function x(t)

So what are we basically doing? **WE ARE WORKING BACKWARDS!!!!! OR FINDING THE ANTI -DERIVATIVE**

Example

An object is moving at velocity with respect to time according to the equation v(t) = 2t.

a) What is the displacement function? Hint: What was the ORIGINAL FUCNTION BEFORE the “derivative? was taken?

b) How FAR did it travel from t = 2s to t = 7s?

$$x(t) = \int v\,dt \rightarrow \int (2t)dt =$$

$$x(t) = t^2$$

These are your LIMITS!

$$x(t) = \int_{t=2}^{t=7} v\,dt \rightarrow \int_{t=2}^{t=7} (2t)\,dt \rightarrow_{t=2}^{t=7} t^2$$

$$7^2 - 2^2 \rightarrow 49 - 4 = \textbf{45 m}$$

You might have noticed that in the above example we had to find the **change(Δ)** over the integral to find the area, that is why we subtract. This might sound confusing. But integration does mean SUM. What we are doing is finding the TOTAL AREA from 0-7 and then the TOTAL AREA from 0-2. Then we can subtract the two numbers to get JUST THE AREA from 2-7.

In summary…

So basically derivatives are used to find **SLOPES** and Integrals are used to find **AREAS**.

$$v = \frac{dx}{dt} \qquad a = \frac{dv}{dt}$$

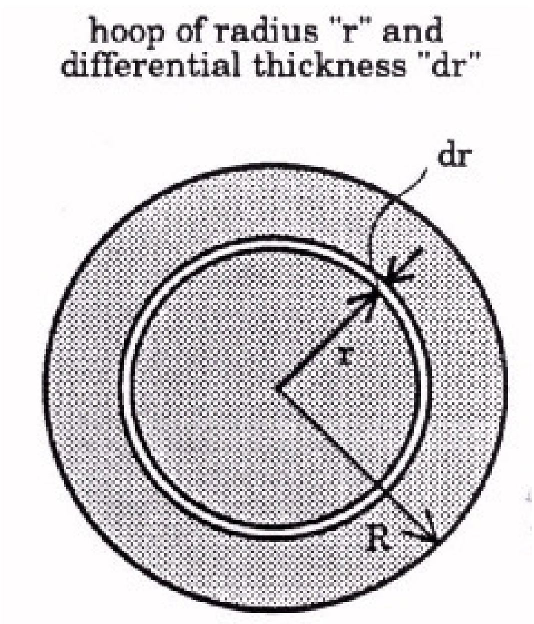
$$x = \int v\,dt \qquad v = \int a\,dt$$

When do I use limits?

There are only TWO things you will be asked to do.

- **DERIVE** – Simply find a function, which do not require limits
- **EVALUATE** – Find the function and solve using a given set of limits.

Example



Here is a simple example of which you may be familiar with:
Assume we know the circumference of a circle is $2\pi r$, where ***r*** is the radius. How can we derive an expression for the area of a circle whose radius is ***R***?

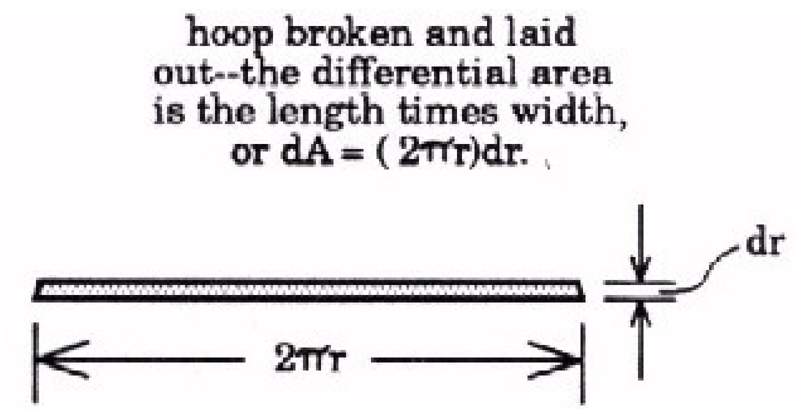
We begin by taking a differential HOOP of radius "*r*" and differential thickness "*dr*" as shown.

If we determine the area of JUST OUR CHOSEN HOOP, we could do the calculation for ALL the possible hoops **inside** the circle.

Having done so, we would then SUM up all of those hoops to find the TOTAL AREA of the circle. The limits are going to be the two extremes, when $r = R$ and when $r = 0$

Example cont'

If we break this hoop and make it flat, we see that it is basically a rectangle with the base equal to the circumference and the height equal to "dr".



$$Area = BasexHeight$$

$$dA = (2\pi r)dr$$

$$A = \int dA = \int_{r=0}^{R} (2\pi r)dr$$

$$= 2\pi \int_{r=0}^{R} (r)dr$$

$$= 2\pi [\frac{r^2}{2}]_{r=0}^{R}$$

$$= 2\pi [(\frac{R^2}{2}) - (0)]$$

$$= \pi R^2$$

Can we cheat? …YES!

Here is the integral equation. Simply take the exponent, add one, then divide by the exponent plus one.

$$\frac{x^{n+1}}{n+1}, n = \text{exponent}$$

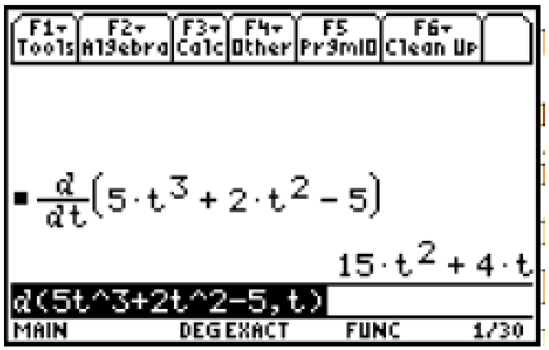
$$3x^2 \rightarrow 3\int x^2 \rightarrow 3\left(\frac{x^{2+1}}{2+1}\right) = x^3$$

$$nx^{n-1} \rightarrow \frac{dx^3}{dt} \rightarrow 3x^2$$

The perfect tool…TI-89!

The TI-89 graphing calculator can do ALL the calculus to truly need to do. Whether you are DERIVING or EVALUATING a function it can help you get the correct answer.

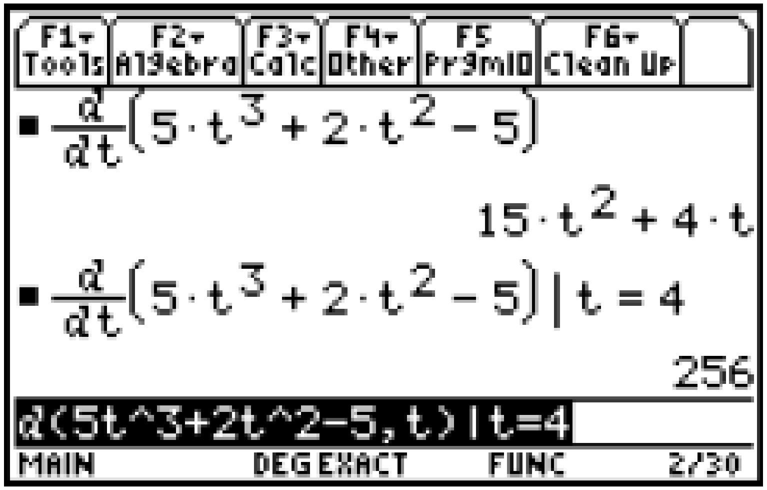
First let me show you how to put a derivative into the calculator.



Using the SECOND key the derivative symbol is just above the number "8" key. A parenthesis will automatically appear. Type in the function you want like $5t^3+2t^2-5$. After typing the function, put a comma after it and tell the calculator "WITH RESPECT TO WHAT" do you want to find the derivative of .

In this case, we want "WITH RESPECT TO TIME or t" . So we place a "t" after the comma and close the parenthesis. Hit enter and you will find the NEW FUNCTION = DERIVATIVE.

The perfect tool…TI-89!



Suppose we want to now EVALUATE this function. In other words, I may want to what the velocity is at exactly t =4 seconds.

Velocity is the derivative of a displacement function! So we first have to derive the new function. Then we have to evaluate it at 4 seconds. All you do is enter a vertical line located next to the "7" key then type in t = 4. As you can see we get a velocity of 256 m/s. That is pretty fast!

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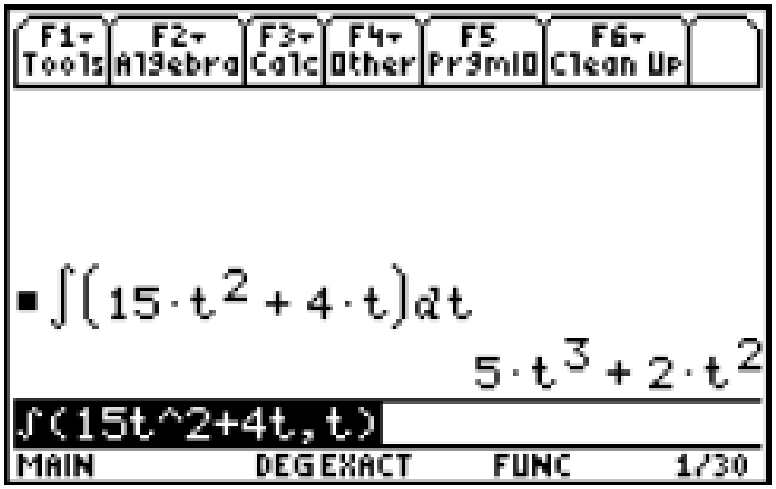
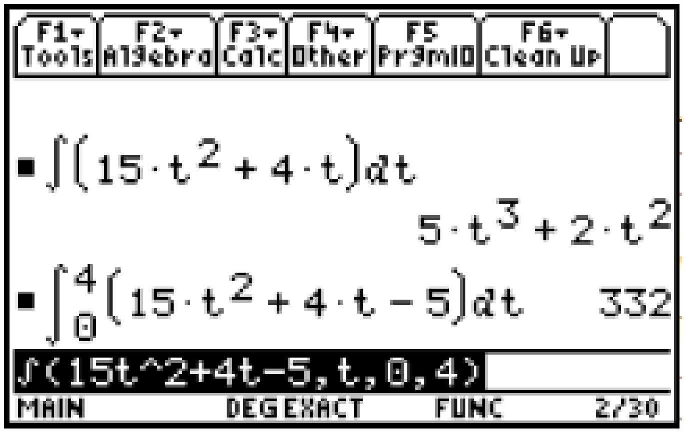


Start by using the second key. The integration symbol is located just ABOVE the "7". It, as well, will come with a parenthesis to begin with. Keep this in mind as you are entering in functions.

Enter the function, then use a comma, then state with respect to what. In this case we have TIME. Remember you can check your answer by taking the derivative of the function to see if you get the original equation.

The perfect tool…TI-89!



Now let's look at evaluating the function. THIS IS CALLED APPLYING THE LIMITS. In other words, over what period are we summing the area. It could be a length of time, a distance, an area, a volume...ANYTHING!

After stating what you are with respect to, enter in the LOWER LIMIT first, then the UPPER LIMIT, then close the parenthesis. So let's say this function was a VELOCITY function. The area under the graph represents DISPLACEMENT. So that means in FOUR SECONDS this object traveled 332 meters.

Example

A particle moving in one dimension has a position function defined as:

$$x(t) = 6t^4 - 2t$$

a) At what point in time does the particle change its direction along the x-axis?

The body will change its direction when it reaches either its maximum or minimum x position. At that point it will reverse its direction. The velocity at the turn around point is ZERO. Thus the velocity function is:

$$v = \frac{dx}{dt} = \frac{d(6t^4 - 2t)}{dt} =$$

$$v = 24t^3 - 2$$

$$0 = 24t^3 - 2$$

$$2 = 24t^3$$

$$t = \sqrt[3]{\frac{2}{24}} = 0.437s$$

Example

b) In what direction is the body traveling when its acceleration is 12 m/s/s?

If we can determine the time at which a = +12 m/s, we can put that time back into our velocity function to determine the velocity of the body at that point of motion. Knowing the velocity (sign and all) will tell us the direction of motion.

$$a = \frac{dv}{dt} = \frac{d(24t^3 - 2)}{dt}$$

$$a = 72t^2$$

$$12 = 72t^2$$

$$t = \sqrt{\frac{12}{72}} = 0.408s$$

$$v = 24t^3 - 2$$

$$v = 24(.408)^3 - 2$$

$$v = -0.37\,m/s$$

The velocity vector is negative, the body must be moving in the negative direction when the acceleration is +12 m/s/s.