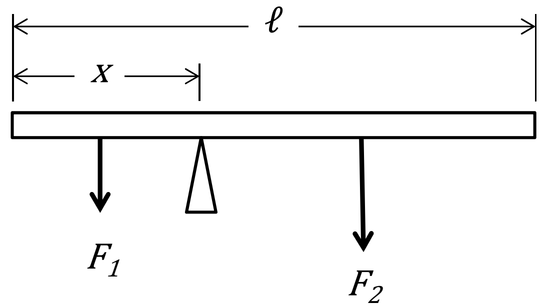
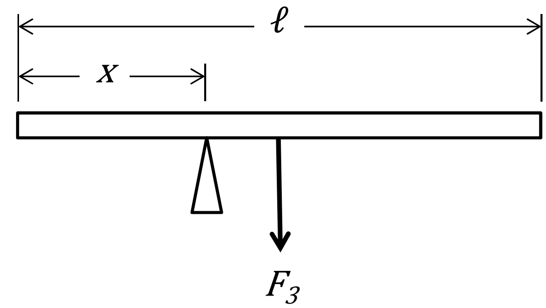
Why can we use the center of a uniform object as the location where the gravitational force acts, when part of the object hangs on each side of the pivot?

I will calculate the torque of an object length ℓ around a fulcrum located a distance x from the left end in two different ways. First, I analyze the situation pictured on the left: I treat each section of the object independently with a force of gravity proportional to the percentage of the mass that is on each side of the fulcrum acts at a distance of half the length of each section. Next, I analyze the situation shown to the right: I use the center of mass of the object to be the location where the entire gravitational force acts.





Analyzing the situation depicted on the left:

F_2 provides a clockwise torque, while F_1 provides a counterclockwise torque. The magnitude of the net torque will be the difference between the two torques.

$$\tau = r_2 F_2 - r_1 F_1$$

F_1 is the gravitational force on the left section. This will be a fraction $\frac{x}{\ell}$ of the entire weight. Similarly, F_2, the gravitational force on the right section, will be the remaining fraction $\frac{\ell - x}{\ell}$ of the entire weight.

$$F_1 = \frac{x}{\ell} mg \qquad F_2 = \frac{\ell - x}{\ell} mg$$

The effective radius on the left will be half of x: $r_1 = \frac{x}{2}$ and the effective radius on the right will be half of the remaining section: $r_2 = \frac{1}{2}(\ell - x)$.

Substituting into the torque equation and simplifying:

$$\tau = r_2 F_2 - r_1 F_1$$

$$\tau = \frac{1}{2}(\ell - x)\left(\frac{\ell - x}{\ell}\right) mg - \left(\frac{x}{2}\right)\left(\frac{x}{\ell}\right) mg$$

$$\tau = \left(\frac{\ell}{2} - \frac{x}{2}\right)\left(\frac{\ell}{\ell} - \frac{x}{\ell}\right) mg - \left(\frac{x}{2}\right)\left(\frac{x}{\ell}\right) mg$$

Factor out mg:

$$\tau = \left(\left(\frac{\ell}{2} - \frac{x}{2}\right)\left(1 - \frac{x}{\ell}\right) - \frac{x^2}{2\ell}\right) mg$$

FOIL left side:

$$\tau = \left(\frac{\ell}{2} - \frac{x\ell}{2\ell} - \frac{x}{2} + \frac{x^2}{2\ell} - \frac{x^2}{2\ell}\right) mg$$

Cancel, regroup:

$$\tau = \left(\frac{\ell}{2} - \frac{x\cancel{\ell}}{2\cancel{\ell}} - \frac{x}{2} + \cancel{\frac{x^2}{2\ell}} - \cancel{\frac{x^2}{2\ell}}\right) mg$$

$$\tau = \left(\frac{\ell}{2} - \frac{x}{2} - \frac{x}{2}\right) mg$$

$$\tau = \left(\frac{\ell}{2} - \frac{\cancel{2}x}{\cancel{2}}\right) mg$$

$$\tau = \left(\frac{\ell}{2} - x\right) mg$$

Analyzing the situation depicted on the right:

There is only one torque, due to $F_3 = mg$ acting at the center of the meterstick $r = \frac{\ell}{2} - x$

$$\tau = r_3 F_3$$

$$\tau = \left(\frac{\ell}{2} - x\right) mg$$

Notice that the two approaches result in the same torque.