## Chapter 4

## Center of Mass and Momentum




1. A system consists of two objects having masses $m_{1}$ and $m_{2}\left(m_{1}<m_{2}\right)$. The objects are connected by a massless string, hung over a pulley as shown above, and then released. When the speed of each object is $v$, the magnitude of the total linear momentum of the system is
(A) $\left(m_{1}+m_{2}\right) v$
(B) $\left(m_{2}-m_{1}\right) v$
(C) $1 / 2\left(m_{1}+m_{2}\right) v$
(D) $1 / 2\left(m_{2}-m_{1}\right) v$
(E) $\mathrm{m}_{2} \mathrm{~V}$

2. Two particles of equal mass $m_{0}$, moving with equal speeds $v_{O}$ along paths inclined at $60^{\circ}$ to the $x$-axis as shown above, collide and stick together. Their velocity after the collision has magnitude
(A) $\frac{v_{0}}{4}$
(B) $\frac{v_{0}}{2}$
(C) $\frac{\sqrt{2} v_{0}}{2}$
(D) $\frac{\sqrt{3} v_{0}}{2}$
(E) $v_{o}$

3. The center of mass of a uniform wire, bent in the shape shown above, is located closest to point
(A) A
(B) B
(C) C
(D) D
(E) E
4. Mass $M_{1}$ is moving with speed $v$ toward stationary mass $M_{2}$. The speed of the center of mass of the system is
(A) $\frac{M_{1}}{M_{2}} v$
(B) $\left(1+\frac{M_{1}}{M_{2}}\right) v$
(C) $\left(1+\frac{M_{2}}{M_{1}}\right) v$
(D) $\left(1-\frac{M_{1}}{M_{2}}\right) v$
(E) $\left(\frac{M_{1}}{M_{1}+M_{2}}\right) v$

## Questions 5-6



A 4-kilogram mass has a speed of 6 meters per second on a horizontal frictionless surface, as shown above. The mass collides head-on and elastically with an identical 4-kilogram mass initially at rest. The second 4-kilogram mass then collides head-on and sticks to a third 4-kilogram mass initially at rest.
5. The final speed of the first 4-kilogram mass is
(A) $0 \mathrm{~m} / \mathrm{s}$
(B) $2 \mathrm{~m} / \mathrm{s}$
(C) $3 \mathrm{~m} / \mathrm{s}$
(D) $4 \mathrm{~m} / \mathrm{s}$
(E) $6 \mathrm{~m} / \mathrm{s}$
6. The final speed of the two 4-kilogram masses that stick together is
(A) $0 \mathrm{~m} / \mathrm{s}$
(B) $2 \mathrm{~m} / \mathrm{s}$
(C) $3 \mathrm{~m} / \mathrm{s}$
(D) $4 \mathrm{~m} / \mathrm{s}$
(E) $6 \mathrm{~m} / \mathrm{s}$
7. A projectile of mass $M_{1}$ is fired horizontally from a spring gun that is initially at rest on a frictionless surface. The combined mass of the gun and projectile is $M_{2}$. If the kinetic energy of the projectile after firing is K , the gun will recoil with a kinetic energy equal to
(A) $K$
(B) $\frac{M_{2}}{M_{1}} K$
(C) $\frac{M_{1}^{2}}{M_{2}^{2}} K$
D) $\frac{M_{1}}{M_{2}-M_{1}} K$
(E) $\sqrt{\frac{M_{1}}{M_{2}-M_{1}}} K$

8. A piece of wire of uniform cross section is bent in the shape shown above. What are the coordinates $(\bar{x}, \bar{y})$ of the center of mass?
(A) $(15 / 14,6 / 7)$
(B) $(6 / 7,6 / 7)$
(C) $(15 / 14,8 / 7)$
(D) $(1,1)$
(E) $(1,6 / 7)$


Figure I


Figure II
9. Two balls are on a frictionless horizontal tabletop. Ball $X$ initially moves at 10 meters per second, as shown in Figure I above. It then collides elastically with identical ball Y. which is initially at rest. After the collision, ball X moves at 6 meters per second along a path at $53^{0}$ to its original direction, as shown in Figure II above. Which of the following diagrams best represents the motion of ball Y after the collision?
(A)

(B)

(C)

(D)

(E)

10. A balloon of mass $M$ is floating motionless in the air. A person of mass less than $M$ is on a rope ladder hanging from the balloon. The person begins to climb the ladder at a uniform speed v relative to the ground. How does the balloon move relative to the ground?
(A) Up with speed v
(B) Up with a speed less than $v$
(C) Down with speed v
(D) Down with a speed less than v
(E) The balloon does not move.
11. If one knows only the constant resultant force acting on an object and the time during which this force acts, one can determine the
(A) change in momentum of the object
(B) change in velocity of the object
(C) change in kinetic energy of the object
(D) mass of the object
(E) acceleration of the object

12. An object of mass $m$ is moving with speed $v_{0}$ to the right on a horizontal frictionless surface, as shown above, when it explodes into two pieces. Subsequently, one piece of mass $2 / 5 \mathrm{~m}$ moves with a speed $\mathrm{v}_{0} / 2$ to the left. The speed of the other piece of the object is
(A) $v_{o} / 2$
(B) $v_{o} / 3$
(C) $7 \mathrm{v}_{\mathrm{o}} / 5$
(D) $3 \mathrm{v}_{\mathrm{o}} / 2$
(E) $2 v_{o}$

13. A 5-kilogram sphere is connected to a 10-kilogram sphere by a rigid rod of negligible mass, as shown above. Which of the five lettered points represents the center of mass of the sphere-rod combination?
(A) A
(B) B
(C) C
(D) D
(E) E

14. The graph above shows the force on an object of mass M as a function of time. For the time interval 0 to 4 s , the total change in the momentum of the object is
(A) $40 \mathrm{~kg} \mathrm{~m} / \mathrm{s}$
(B) $20 \mathrm{~kg} \mathrm{~m} / \mathrm{s}$
(C) $0 \mathrm{~kg} \mathrm{~m} / \mathrm{s}$
(D) $-20 \mathrm{~kg} \mathrm{~m} / \mathrm{s}$
(E) indeterminable unless the mass M of the object is known


Top View
15. As shown in the top view above, a disc of mass $m$ is moving horizontally to the right with speed $v$ on a table with negligible friction when it collides with a second disc of mass 2 m The second disc is moving horizontally to the right with speed $v / 2$ at the moment of impact The two discs stick together upon impact The speed of the composite body immediately after the collision is
(A) $v / 3$
(B) $v / 2$
(C) $2 \mathrm{v} / 3$
(D) $3 \mathrm{v} / 2$
(E) 2 v
16. Two people are initially standing still on frictionless ice. They push on each other so that one person, of mass 120 kg , moves to the left at $2 \mathrm{~m} / \mathrm{s}$, while the other person, of mass 80 kg , moves to the right at $3 \mathrm{~m} / \mathrm{s}$. What is the velocity of their center of mass?
(A) Zero
(B) $0.5 \mathrm{~m} / \mathrm{s}$ to the left
(C) $1 \mathrm{~m} / \mathrm{s}$ to the right
(D) $2.4 \mathrm{~m} / \mathrm{s}$ to the left
(E) $2.5 \mathrm{~m} / \mathrm{s}$ to the right
17. An object having an initial momentum that may be represented by the vector above strikes an object that is initially at rest. Which of the following sets of vectors may represent the momenta of the two objects after the collision?
(A)

(B)

(C)
(D)

(E)

18. A 2 kg ball collides with the floor at an angle $\theta$ and rebounds at the same angle and speed as shown above. Which of the following vectors represents the impulse exerted on the ball by the floor?
(A)
(B)

(C)
(D)

(E) $\uparrow$
19. The momentum $p$ of a moving object as a function of time $t$ is given by the expression $p=k t^{3}$, where $k$ is a constant. The force causing this motion is given by the expression
(A) $3 k t^{2}$
(B) $3 k t^{2} / 2$
(C) $k t^{2} / 3$
(D) $k t^{4}$
(E) $k t^{4} / 4$


Two pucks moving on a frictionless air table are about to collide, as shown above. The 1.5 kg puck is moving directly east at $2.0 \mathrm{~m} / \mathrm{s}$. The 4.0 kg puck is moving directly north at $1.0 \mathrm{~m} / \mathrm{s}$.
20. What is the total kinetic energy of the two-puck system before the collision?
(A) $\sqrt{13} \mathrm{~J}$
(B) 5.0 J
(C)
7.0 J
(D) 10 J
(E) 11 J
21. What is the magnitude of the total momentum of the two-puck system after the collision?
(A) $1.0 \mathrm{~kg} \cdot \mathrm{~m} / \mathrm{s}$
(B) $3.5 \mathrm{~kg} \cdot \mathrm{~m} / \mathrm{s}$
(C) $5.0 \mathrm{~kg} \cdot \mathrm{~m} / \mathrm{s}$
(D) $7.0 \mathrm{~kg} \cdot \mathrm{~m} / \mathrm{s}$
(E) $5.5 \sqrt{5} \mathrm{~kg} \cdot \mathrm{~m} / \mathrm{s}$

22. As shown above, two students sit at opposite ends of a boat that is initially at rest. The student in the front throws a heavy ball to the student in the back. What is the motion of the boat at the time immediately after the ball is thrown and, later, after the ball is caught? (Assume that air and water friction are negligible.)

Immediately
After the Throw
(A) Boat moves forward
(B) Boat moves forward
(C) Boat moves forward
(D) Boat moves backward
(E) Boat moves backward

After the Catch
Boat moves forward
Boat moves backward
Boat does not move
Boat does not move
Boat moves forward
23. A person holds a portable fire extinguisher that ejects 1.0 kg of water per second horizontally at a speed of 6.0 $\mathrm{m} / \mathrm{s}$. What horizontal force in newtons must the person exert on the extinguisher in order to prevent it from accelerating?
(A) 0 N
(B) 6 N
(C) 10 N
(D) 18 N
(E) 36 N

24. A person is standing at one end of a uniform raft of length $L$ that is floating motionless on water, as shown above. The center of mass of the person-raft system is a distance $d$ from the center of the raft. The person then walks to the other end of the raft. If friction between the raft and the water is negligible, how far does the raft move relative to the water?
(A) $\frac{L}{2}$
(B) $L$
(C) $\frac{d}{2}$
(D) $d$
(E) $2 d$
25. Objects 1 and 2 have the same momentum. Object 1 can have more kinetic energy than object 2 if, compared with object 2 , it
(A) has more mass
(B) has the same mass
(C) is moving at the same speed
(D) is moving slower
(E) is moving faster
26. A 5 kg object is propelled from rest at time $t=0$ by a net force $\mathbf{F}$ that always acts in the same direction. The magnitude of $\mathbf{F}$ in newtons is given as a function of $t$ in seconds by $F=0.5 t$. What is the speed of the object at $t=4 \mathrm{~s}$ ?
(A) $0.5 \mathrm{~m} / \mathrm{s}$
(B) $0.8 \mathrm{~m} / \mathrm{s}$
(C) $2.0 \mathrm{~m} / \mathrm{s}$
(D) $4.0 \mathrm{~m} / \mathrm{s}$
(E) $8.0 \mathrm{~m} / \mathrm{s}$

27. Two balls with masses $m$ and $2 m$ approach each other with equal speeds $v$ on a horizontal frictionless table, as shown in the top view above. Which of the following shows possible velocities of the balls for a time soon after the balls collide?
(A)


(B)
(C)

(D)

(E)


28. Three identical disks are initially at rest on a frictionless, horizontal table with their edges touching to form a triangle, as shown in the top view above. An explosion occurs within the triangle, propelling the disks horizontally along the surface. Which of the following diagrams shows a possible position of the disks at a later time? (In these diagrams, the triangle is shown in its original position.)



1976M3. A bullet of mass $m$ and velocity $v_{o}$ is fired toward a block of thickness $L_{o}$ and mass $M$. The block is initially at rest on a frictionless surface. The bullet emerges from the block with velocity $\mathrm{v}_{0} / 3$.
a. Determine the final speed of block M.
b. If, instead, the block is held fixed and not allowed to slide, the bullet emerges from the block with a speed $\mathrm{v}_{\mathrm{o}} / 2$. Determine the loss of kinetic energy of the bullet
c. Assume that the retarding force that the block material exerts on the bullet is constant. In terms of $\mathrm{L}_{0}$, what minimum thickness L should a fixed block of similar material have in order to stop the bullet?
d. When the block is held fixed, the bullet emerges from the block with a greater speed than when the block is free to move. Explain.


1979M1. A ball of mass $m$ is released from rest at a distance $h$ above a frictionless plane inclined at an angle of $45^{\circ}$ to the horizontal as shown above. The ball bounces elastically off the plane at point $\mathrm{P}_{1}$ and strikes the plane again at point $P_{2}$. In terms of $g$ and $h$ determine each of the following quantities:
a. The velocity (a vector) of the ball just after it first bounces off the plane at $\mathrm{P}_{1}$.
b. The time the ball is in flight between points $\mathrm{P}_{1}$ and $\mathrm{P}_{2}$.
c. The distance $L$ along the plane from $P_{1}$ to $P_{2}$.
d. The speed of the ball just before it strikes the plane at $\mathrm{P}_{2}$.


1979M2. A ferryboat of mass $M_{1}=2.0 \times 10^{5}$ kilograms moves toward a docking bumper of mass $M_{2}$ that is attached to a shock absorber. Shown below is a speed v vs. time t graph of the ferryboat from the time it cuts off its engines to the time it first comes to rest after colliding with the bumper.
At the instant it hits the bumper, $\mathrm{t}=0$ and $\mathrm{v}=3$ meters per second.

a. After colliding inelastically with the bumper, the ferryboat and bumper move together with an initial speed of 2 meters per second. Calculate the mass of the bumper $\mathrm{M}_{2}$.
b. After colliding, the ferryboat and bumper move with a speed given by the expression $v=2 e^{-4 t}$. Although the boat never comes precisely to rest, it travels only a finite distance. Calculate that distance.
c. While the ferryboat was being slowed by water resistance before hitting the bumper, its speed was given by $1 / v=1 / 3+\beta t$, where $\beta$ is a constant. Find an expression for the retarding force of the water on the boat as a function of speed.


1980M2. A block of mass $m$ slides at velocity $\mathrm{v}_{\mathrm{o}}$ across a horizontal frictionless surface toward a large curved movable ramp of mass 3 m as shown in Figure 1. The ramp, initially at rest, also can move without friction and has a smooth circular frictionless face up which the block can easily slide. When the block slides up the ramp, it momentarily reaches a maximum height as shown in Figure II and then slides back down the frictionless face to the horizontal surface as shown in Figure III.
a. Find the velocity $\mathrm{v}_{1}$ of the moving ramp at the instant the block reaches its maximum height.
b. To what maximum height $h$ does the center of mass of the block rise above its original height?
c. Determine the final speed $v_{f}$ of the ramp and the final speed $v^{\prime}$ of the block after the block returns to the level surface. State whether the block is moving to the right or to the left.

1985M1. A projectile is launched from the top of a cliff above level ground. At launch the projectile is 35 meters above the base of the cliff and has a velocity of 50 meters per second at an angle $37^{\circ}$ with the horizontal. Air resistance is negligible. Consider the following two cases and use $g=10 \mathrm{~m} / \mathrm{s}^{2}, \sin 37^{\circ}=0.60$, and $\cos 37^{\circ}=$ 0.80 .

Case I: The projectile follows the path shown by the curved line in the following diagram.

a. Calculate the total time from launch until the projectile hits the ground at point C .
b. Calculate the horizontal distance R that the projectile travels before it hits the ground.
c. Calculate the speed of the projectile at points A, B and C.


Case II: A small internal charge explodes at point B in the above diagram, causing the projectile to separate into two parts of masses 6 kilograms and 10 kilograms. The explosive force on each part is horizontal and in the plane of the trajectory. The 6-kilogram mass strikes the ground at point D , located 30 meters beyond point C , where the projectile would have landed had it not exploded The 10 -kilogram mass strikes the ground at point E .
d. Calculate the distance x from C to $E$.


1991M1. A small block of mass 2 m initially rests on a track at the bottom of the circular, vertical loop-the-loop shown above, which has a radius r . The surface contact between the block and the loop is frictionless. A bullet of mass $m$ strikes the block horizontally with initial speed $v_{o}$ and remains embedded in the block as the block and bullet circle the loop. Determine each of the following in terms of $m, v_{0} r$, and $g$.
a. The speed of the block and bullet immediately after impact
b. The kinetic energy of the block and bullet when they reach point $P$ on the loop
c. The minimum initial speed $\mathrm{v}_{\text {min }}$ of the bullet if the block and bullet are to successfully execute a complete circuit of the loop


1991M3. The two blocks I and II shown above have masses m and 2 m respectively. Block II has an ideal massless spring attached to one side. When block I is placed on the spring as shown. the spring is compressed a distance D at equilibrium. Express your answer to all parts of the question in terms of the given quantities and physical constants.
a. Determine the spring constant of the spring


Later the two blocks are on a frictionless, horizontal surface. Block II is stationary and block I approaches with a speed $v_{0}$, as shown above.
b. The spring compression is a maximum when the blocks have the same velocity. Briefly explain why this is so.
c. Determine the maximum compression of the spring during the collision.
d. Determine the velocity of block II after the collision when block I has again separated from the spring.


1992M1. A ball of mass 9 m is dropped from rest from a height $\mathrm{H}=5.0$ meters above the ground, as shown above on the left. It undergoes a perfectly elastic collision with the ground and rebounds. At the instant that the ball rebounds, a small blob of clay of mass $m$ is released from rest from the original height $H$, directly above the ball, as shown above on the right. The clay blob, which is descending, eventually collides with the ball, which is ascending. Assume that $g=10 \mathrm{~m} / \mathrm{s}^{2}$, that air resistance is negligible, and that the collision process takes negligible time.
a. Determine the speed of the ball immediately before it hits the ground.
b. Determine the time after the release of the clay blob at which the collision takes place.
c. Determine the height above the ground at which the collision takes place.
d. Determine the speeds of the ball and the clay blob immediately before the collision.
e. If the ball and the clay blob stick together on impact, what is the magnitude and direction of their velocity immediately after the collision?


Figure I


1993M1. A massless spring with force constant $k=400$ newtons per meter is fastened at its left end to a vertical wall, as shown in Figure 1. Initially, block $C$ (mass $m_{c}=4.0$ kilograms) and block $D$ (mass $m_{D}=2.0$ kilograms) rest on a horizontal surface with block $C$ in contact with the spring (but not compressing it) and with block $D$ in contact with block $C$. Block $C$ is then moved to the left, compressing the spring a distance of 0.50 meter, and held in place while block $D$ remains at rest as shown in Figure 11. (Use $g=10 \mathrm{~m} / \mathrm{s}^{2}$.)
a. Determine the elastic energy stored in the compressed spring.

Block $C$ is then released and accelerates to the right, toward block $D$. The surface is rough and the coefficient of friction between each block and the surface is $\mu=0.4$. The two blocks collide instantaneously, stick together, and move to the right. Remember that the spring is not attached to block $C$. Determine each of the following.
b. The speed $\mathrm{v}_{\mathrm{c}}$ of block $C$ just before it collides with block $D$
c. The speed $\mathrm{v}_{\mathrm{f}}$ blocks $C$ and $D$ just after they collide
d. The horizontal distance the blocks move before coming to rest


1994M1. A 2-kilogram block and an 8-kilogram block are both attached to an ideal spring ( for which $\mathrm{k}=200 \mathrm{~N} / \mathrm{m}$ ) and both are initially at rest on a horizontal frictionless surface, as shown in the diagram above.
In an initial experiment, a 100-gram $(0.1 \mathrm{~kg})$ ball of clay is thrown at the 2-kilogram block. The clay is moving horizontally with speed v when it hits and sticks to the block. The 8 -kilogram block is held still by a removable stop. As a result, the spring compresses a maximum distance of 0.4 meters.
a. Calculate the energy stored in the spring at maximum compression.
b. Calculate the speed of the clay ball and 2-kilogram block immediately after the clay sticks to the block but before the spring compresses significantly.
c. Calculate the initial speed $v$ of the clay.

In a second experiment, an identical ball of clay is thrown at another identical 2-kilogram block, but this time the stop is removed so that the 8-kilogram block is free to move.
d. State whether the maximum compression of the spring will be greater than, equal to, or less than 0.4 meter. Explain briefly.
e. State the principle or principles that can be used to calculate the velocity of the 8-kilogram block at the instant that the spring regains its original length. Write the appropriate equation(s) and show the numerical substitutions, but do not solve for the velocity.


## Note: Figure not drawn to scale.

1995M1. A 5-kilogram ball initially rests at the edge of a 2-meter-long, 1.2-meter-high frictionless table, as shown above. A hard plastic cube of mass 0.5 kilogram slides across the table at a speed of 26 meters per second and strikes the ball, causing the ball to leave the table in the direction in which the cube was moving. The figure below shows a graph of the force exerted on the ball by the cube as a function of time.

a. Determine the total impulse given to the ball.
b. Determine the horizontal velocity of the ball immediately after the collision.
c. Determine the following for the cube immediately after the collision.
i. Its speed
ii. Its direction of travel (right or left), if moving
d. Determine the kinetic energy dissipated in the collision.
e. Determine the distance between the two points of impact of the objects with the floor.


1997M2. An open-top railroad car (initially empty and of mass $M_{o}$ ) rolls with negligible friction along a straight horizontal track and passes under the spout of a sand conveyor. When the car is under the conveyor, sand is dispensed from the conveyor in a narrow stream at a steady rate $\Delta \mathrm{M} / \Delta \mathrm{t}=\mathrm{C}$ and falls vertically from an average height $h$ above the floor of the railroad car. The car has initial speed $v_{o}$ and sand is filling it from time $t=0$ to $t$ $=\mathrm{T}$. Express your answers to the following in terms of the given quantities and $g$.
a. Determine the mass M of the car plus the sand that it catches as a function of time t for $0<\mathrm{t}<\mathrm{T}$.
b. Determine the speed $v$ of the car as a function of time $t$ for $0<t<T$.
c. i. Determine the initial kinetic energy $K_{i}$ of the empty car.
ii. Determine the final kinetic energy $\mathrm{K}_{\mathrm{f}}$ of the car and its load.
iii. Is kinetic energy conserved? Explain why or why not.
d. Determine expressions for the normal force exerted on the car by the tracks at the following times.
i. Before $\mathrm{t}=0$
ii. For $0<t<T$
iii. After $\mathrm{t}=\mathrm{T}$


## Air Track

1998M1. Two gliders move freely on an air track with negligible friction, as shown above. Glider A has a mass of 0.90 kg and glider B has a mass of 0.60 kg . Initially, glider A moves toward glider B, which is at rest. A spring of negligible mass is attached to the right side of glider A. Strobe photography is used to record successive positions of glider A at 0.10 s intervals over a total time of 2.00 s , during which time it collides with glider B.

The following diagram represents the data for the motion of glider A. Positions of glider A at the end of each 0.10 s interval are indicated by the symbol A against a metric ruler. The total elapsed time $t$ after each 0.50 s is also indicated.
a. Determine the average speed of glider A for the following time intervals.

i. 0.10 s to 0.30 s
ii. 0.90 s to 1.10 s
iii. 1.70 s to 1.90 s
b. On the axes below, sketch a graph, consistent with the data above, of the speed of glider A as a function of time t for the 2.00 s interval.

c. i. Use the data to calculate the speed of glider B immediately after it separates from the spring. ii. On the axes below, sketch a graph of the speed of glider B as a function of time $t$.


A graph of the total kinetic energy K for the two-glider system over the 2.00 s interval has the following shape. $K_{o}$ is the total kinetic energy of the system at time $t=0$.

d. i. Is the collision elastic? Justify your answer.
ii. Briefly explain why there is a minimum in the kinetic energy curve at $\mathrm{t}=1.00 \mathrm{~s}$.


1999M1 In a laboratory experiment, you wish to determine the initial speed of a dart just after it leaves a dart gun.
The dart, of mass $m$, is fired with the gun very close to a wooden block of mass $\mathrm{M}_{0}$ which hangs from a cord of length $l$ and negligible mass, as shown above. Assume the size of the block is negligible compared to $l$, and the dart is moving horizontally when it hits the left side of the block at its center and becomes embedded in it. The block swings up to a maximum angle from the vertical. Express your answers to the following in terms of m , $\mathrm{M}_{0}, l, \theta_{\text {max }}$, and g .
a. Determine the speed $\mathrm{v}_{0}$ of the dart immediately before it strikes the block.
b. The dart and block subsequently swing as a pendulum. Determine the tension in the cord when it returns to the lowest point of the swing.
c. At your lab table you have only the following additional equipment.

| Meter stick | Stopwatch | Set of known masses |
| :--- | :--- | :--- |
| Protractor | 5 m of string | Five more blocks of mass $\mathrm{M}_{0}$ |
| Spring |  |  |

Without destroying or disassembling any of this equipment, design another practical method for determining the speed of the dart just after it leaves the gun. Indicate the measurements you would take, and how the speed could be determined from these measurements.
d. The dart is now shot into a block of wood that is fixed in place. The block exerts a force F on the dart that is proportional to the dart's velocity v and in the opposite direction, that is $\mathrm{F}=-\mathrm{bv}$, where b is a constant. Derive an expression for the distance $L$ that the dart penetrates into the block, in terms of $\mathrm{m}, \mathrm{v}_{0}$, and b .




2001M1. A motion sensor and a force sensor record the motion of a cart along a track, as shown above. The cart is given a push so that it moves toward the force sensor and then collides with it. The two sensors record the values shown in the above graphs.
a. Determine the cart's average acceleration between $t=0.33 \mathrm{~s}$ and $\mathrm{t}=0.37 \mathrm{~s}$.
b. Determine the magnitude of the change in the cart's momentum during the collision.
c. Determine the mass of the cart.
d. Determine the energy lost in the collision between the force sensor and the cart


2004 M 1 . A rope of length L is attached to a support at point C . A person of mass $\mathrm{m}_{1}$ sits on a ledge at position A holding the other end of the rope so that it is horizontal and taut, as shown above. The person then drops off the ledge and swings down on the rope toward position $B$ on a lower ledge where an object of mass $m_{2}$ is at rest. At position $B$ the person grabs hold of the object and simultaneously lets go of the rope. The person and object then land together in the lake at point D , which is a vertical distance L below position B . Air resistance and the mass of the rope are negligible. Derive expressions for each of the following in terms of $m_{1}, m_{2}, L$, and $g$.
a. The speed of the person just before the collision with the object
b. The tension in the rope just before the collision with the object
c. The speed of the person and object just after the collision
d. The ratio of the kinetic energy of the person-object system before the collision to the kinetic energy after the collision
e. The total horizontal displacement $x$ of the person from position $A$ until the person and object land in the water at point $D$.


2010M3. A skier of mass $m$ will be pulled up a hill by a rope, as shown above. The magnitude of the acceleration of the skier as a function of time $t$ can be modeled by the equations

$$
\begin{aligned}
a & =a_{\max } \sin (\pi t / T) & & (0<t<T) \\
& =0 & & (t \leq T) .
\end{aligned}
$$

where $a_{\text {max }}$ and $T$ are constants. The hill is inclined at an angle $\theta$ above the horizontal, and friction between the skis and the snow is negligible. Express your answers in terms of given quantities and fundamental constants.
a. Derive an expression for the velocity of the skier as a function of time during the acceleration. Assume the skier starts from rest.
b. Derive an expression for the work done by the net force on the skier from rest until terminal speed is reached.
c. Determine the magnitude of the force exerted by the rope on the skier at terminal speed.
d. Derive an expression for the total impulse imparted to the skier during the acceleration.

## Solution

Answer
B

B

B located on the $y$ axis. Then the center of mass of the assembly is

$$
\left(x_{c m}, y_{c m}\right)=\frac{m\left(0, \frac{1}{2} L\right)+m\left(\frac{1}{2} L, 0\right)+m\left(L, \frac{1}{2} L\right)}{m+m+m}=\frac{1}{3}\left(\frac{3}{2} L, L\right)=\left(\frac{1}{2} L, \frac{1}{3} L\right)
$$

Point B is above $1 / 4 \mathrm{~L}$, but below $1 / 2 \mathrm{~L}$, so it's the best answer. (All the points are located at $\mathrm{x}=1 / 2 \mathrm{~L}$, which is obviously correct by symmetry.)
4. The definition of the center of mass velocity is the total momentum divided by the total mass
5. When equal masses collide elastically head-on, they exchange velocities
6. In a perfectly inelastic (sticking) collision: $\mathrm{m}_{1} \mathrm{v}_{1}+\mathrm{m}_{2} \mathrm{v}_{2}=\left(\mathrm{m}_{1}+\mathrm{m}_{2}\right) \mathrm{v}^{\prime}$
7. First use the given kinetic energy of mass M1 to determine the projectile speed after.
$K=1 / 2 M_{1} v_{1 f}^{2} \ldots v_{1 f}=\sqrt{ }\left(2 K / M_{1}\right)$. Now solve the explosion problem with $p_{\text {before }}=0=p_{\text {after }}$.
Note that the mass of the gun is $\mathrm{M}_{2}-\mathrm{M}_{1}$ since $\mathrm{M}_{2}$ was given as the total mass.
$0=M_{1} v_{1 f}+\left(M_{2}-M_{1}\right) v_{2 f} \ldots$ now sub in from above for $v_{1 f}$.
$\mathrm{M}_{1}\left(\sqrt{ }\left(2 \mathrm{~K} / \mathrm{M}_{1}\right)\right)=-\left(\mathrm{M}_{2}-\mathrm{M}_{1}\right) \mathrm{v}_{2 \mathrm{f}}$ and find $\mathrm{v}_{2 \mathrm{f}} \ldots \mathrm{v}_{2 \mathrm{f}}=-\mathrm{M}_{1}\left(\sqrt{ }\left(2 K / M_{1}\right)\right) /\left(\mathrm{M}_{2}-\mathrm{M}_{1}\right)$.
Now sub this into $K_{2}=1 / 2\left(M_{2}-M_{1}\right) v_{2 f}^{2}$ and simplify
8. For a complete square loop, the center of mass is centered within the shape. With the upper left segment removed, the center of mass will shift slightly toward the lower right. That is, the new coordinates must be $\mathrm{x}>1, \mathrm{y}<1$
9. Since there is no y momentum before, there cannot be any net y momentum after. The balls have equal masses so you need the $y$ velocities of each ball to be equal after to cancel out the momenta. By inspection, looking at the given velocities and angles and without doing any math, the only one that could possibly make equal y velocities is choice D
10. Since the only forces on the system are between parts of the system, the center of mass must remain stationary. As the smaller mass moves in one direction, the larger mass compensates by moving (a smaller amount) in the opposite direction
11. Definition. $J_{\text {net }}=\Delta p \quad F_{\text {net }} t=\Delta p$
12. Explosion with initial momentum. $\mathrm{p}_{\text {before }}=\mathrm{p}_{\text {after }} \quad \mathrm{mv}_{\mathrm{o}}=\mathrm{m}_{\mathrm{a}} \mathrm{v}_{\mathrm{af}}+\mathrm{m}_{\mathrm{b}} \mathrm{v}_{\mathrm{bf}}$ $\mathrm{mv}_{\mathrm{o}}=(2 / 5 \mathrm{~m})\left(-\mathrm{v}_{\mathrm{o}} / 2\right)+(3 / 5 \mathrm{~m})\left(\mathrm{v}_{\mathrm{bf}}\right) \ldots$ solve for $\mathrm{v}_{\mathrm{bf}}$
13. Let A be the origin. $\mathrm{x}_{\mathrm{cm}}=[(10 \mathrm{~kg})(0)+(5 \mathrm{~kg})(\mathrm{L})] /(10 \mathrm{~kg}+5 \mathrm{~kg})=\mathrm{L} / 3$
14. The area of the Ft graph is the impulse which determines the momentum change. Since the net impulse is zero, there will be zero total momentum change.
15. Perfect inelastic collision. $\mathrm{m}_{1} \mathrm{v}_{1 \mathrm{i}}+\mathrm{m}_{2} \mathrm{v}_{2 \mathrm{i}}=\mathrm{m}_{\text {tot }}\left(\mathrm{v}_{\mathrm{f}}\right) \ldots(\mathrm{m})(\mathrm{v})+(2 \mathrm{~m})(\mathrm{v} / 2)=(3 \mathrm{~m}) \mathrm{v}_{\mathrm{f}}$

E
A
C

D

A

D

D

A
E

B

C

C
16. As the system started at rest and the only forces on the system are between parts of the system, the center of mass must remain stationary
17. The total momentum vector before must match the total momentum vector after. Only choice E has a possibility of a resultant that matches the initial vector.
18. Since the angle and speed are the same, the $x$ component velocity has been unchanged which means there could not have been any x direction momentum change. The y direction velocity was reversed so there must have been an upwards y impulse to change and reverse the velocity.
19. $\mathrm{F}=\mathrm{dp} / \mathrm{dt}$
20. Simply add the energies $1 / 2(1.5)(2)^{2}+1 / 2(4)(1)^{2}$
21. Total momentum before must equal total momentum after. Before, there is an x momentum of $(2)(1.5)=3$ and a $y$ momentum of $(4)(1)=4$ giving a total resultant momentum before using the Pythagorean theorem of 5. The total after must also be 5 .
22. As the system started at rest and the only forces on the system are between parts of the system, the center of mass must remain stationary. While the ball is in motion toward the back of the boat, the boat will move slightly forward. Once the ball is stationary, so is the boat.
23. $\mathrm{F}=\mathrm{dp} / \mathrm{dt}=(\mathrm{dm} / \mathrm{dt}) \Delta \mathrm{v}=(1 \mathrm{~kg} / \mathrm{s})(6 \mathrm{~m} / \mathrm{s})$
24. As the system started at rest and the only forces on the system are between parts of the system, the center of mass of the system must remain stationary. When the person walks to the other side of the boat, it merely flips the system to its mirror image, there the center of the raft is now a distance $d$ to the right of the center of mass of the system. In effect, the person and the center of the raft switch sides.
25. $\quad \mathrm{K}=\mathrm{p}^{2} / 2 \mathrm{~m}$, for $\mathrm{K}_{1}>\mathrm{K}_{2}$ then $\mathrm{p}_{1}{ }^{2} / 2 \mathrm{~m}_{1}>\mathrm{p}_{2}{ }^{2} / 2 \mathrm{~m}_{2}$ but since $\mathrm{p}_{1}=\mathrm{p}_{2}, \mathrm{~m}_{2}>\mathrm{m}_{1}$ and if $\mathrm{m}_{1}<\mathrm{m}_{2}$ with the same momentum then $v_{1}>v_{2}$. Alternately, $K=1 / 2 m v^{2}=1 / 2 p v$ so if $K_{1}>K_{2}$ then $p v_{1}>p v_{2}$
26. $\int F d t=m \Delta v$
27. Inspecting the initial approach diagram, it is clear the system has a net monetum to the left (with no vertical momentum). After the collision, the system must still have a momentum to the left with no vertical component. Only one diagram illustrated this possibility.
28. As the system started at rest and the only forces on the system are between parts of the system, the center of mass of the system must remain stationary. E is the only arrangement that places the center of mass within the initial triangle.

## 1976M3

a. $\quad \mathrm{p}_{\mathrm{i}}=\mathrm{p}_{\mathrm{f}}$
$\mathrm{mv}_{0}=\mathrm{mv}_{0} / 3+\mathrm{MV}$
$\mathrm{V}=(2 / 3) \mathrm{mv}_{0} / \mathrm{M}$
b. $\quad \Delta \mathrm{K}=1 / 2 \mathrm{mv}_{\mathrm{f}}^{2}-1 / 2 \mathrm{mv}_{\mathrm{i}}^{2}=1 / 2 \mathrm{~m}\left(\mathrm{v}_{0} / 2\right)^{2}-1 / 2 \mathrm{mv}_{0}{ }^{2}=-(3 / 8) \mathrm{mv}_{0}{ }^{2}$
c. The work done on the bullet by a constant force $\mathrm{W}=-\mathrm{FL}=\Delta \mathrm{K}$

This constant force can be calculated using the results above: $-\mathrm{FL}_{0}=-(3 / 8) \mathrm{mv}_{0}{ }^{2}$
$\mathrm{F}=3 \mathrm{mv}_{0}{ }^{2} / 8 \mathrm{~L}_{0}$
To stop the bullet, $\Delta \mathrm{K}=-1 / 2 \mathrm{mv}_{0}{ }^{2}$
$-\mathrm{FL}=-\left(3 \mathrm{mv}_{0}{ }^{2} / 8 \mathrm{~L}_{0}\right) \mathrm{L}=-1 / 2 \mathrm{mv}_{0}{ }^{2}$
This gives $\mathrm{L}=4 \mathrm{~L}_{0} / 3$
d. When the block is free to move, the constant force acts over a greater distance, hence more kinetic energy will be lost. Or consider the block will carry off some kinetic energy when it is free to move.

## 1979M1

a. $\quad \mathrm{U}=\mathrm{K} ; \mathrm{mgh}=1 / 2 \mathrm{mv}^{2}$ gives $\mathrm{v}=(2 \mathrm{gh})^{1 / 2}$, since the collision is elastic this is the same speed after the collision.

The direction is horizontally to the right due to the incline's $45^{\circ}$ angle.
b. From $P_{1}$ to $P_{2}$ the ball maintains a horizontal speed of $(2 \mathrm{gh})^{1 / 2}$ and travels a horizontal distance of $L / \sqrt{ } 2$
$\mathrm{d}_{\mathrm{x}}=\mathrm{v}_{\mathrm{x}} \mathrm{t}$ gives $\mathrm{L} / \sqrt{ } 2=(2 \mathrm{gh})^{1 / 2} \mathrm{t}$
vertically, $d_{y}=L / \sqrt{ } 2=1 / 2$ gt $^{2}$
Equating the two distances and solving for $t$ gives $t=(8 h / g)^{1 / 2}$
c. Substitution of $t$ back into either distance expression gives $L=4 \sqrt{ } 2 h$
d. $\quad \mathrm{U}=\mathrm{K} ; \mathrm{mgh}+\mathrm{mgL} / \sqrt{ } 2=1 / 2 \mathrm{mv}_{2}{ }^{2}$
$\mathrm{mgh}+4 \mathrm{mgh}=1 / 2 \mathrm{mv}_{2}{ }^{2}$
$\mathrm{v}_{2}=(10 \mathrm{gh})^{1 / 2}$

## 1979M2

a. $\quad \mathrm{M}_{1} \mathrm{v}_{\mathrm{i}}=\left(\mathrm{M}_{1}+\mathrm{M}_{2}\right) \mathrm{v}_{\mathrm{f}}$
substituting given values gives $\mathrm{M}_{2}=10^{5} \mathrm{~kg}$
b.

$$
x=\int v d t
$$

$x=\int_{0}^{\infty} 2 e^{-4 t} d t=-\left.\frac{1}{2} e^{-4 t}\right|_{0} ^{\infty}=0-\left(-\frac{1}{2}\right)=0.5 \mathrm{~m}$
c. $\quad \mathrm{F}=\mathrm{m}(\mathrm{dv} / \mathrm{dt})=\mathrm{m}(\mathrm{d} / \mathrm{dt})(1 / 3+\beta \mathrm{t})^{-1}$
$=-\mathrm{m} \beta(1 / 3+\beta \mathrm{t})^{-2}$
$=-\mathrm{m} \beta \mathrm{v}^{2}$

## 1980M2

a. $\quad \mathrm{p}_{\mathrm{i}}=\mathrm{p}_{\mathrm{f}}$
$\mathrm{mv}_{0}=\mathrm{mv}_{1}+3 \mathrm{mv}_{1}$ (when the block reaches maximum height, they are traveling the same speed)
$\mathrm{v}_{1}=\mathrm{v}_{0} / 4$
b. Conservation of energy: $1 / 2 \mathrm{mv}_{0}{ }^{2}=1 / 2(\mathrm{~m}+3 \mathrm{~m}) \mathrm{v}_{1}{ }^{2}+\mathrm{mgh}$
$\mathrm{h}=3 \mathrm{v}_{0}{ }^{2} / 8 \mathrm{~g}$
c. This is an elastic collision: $\mathrm{mv}_{0}=\mathrm{mv}^{\prime}+3 \mathrm{mv}_{\mathrm{f}}$ and $\mathrm{v}_{0}=\mathrm{v}_{\mathrm{f}}-\mathrm{v}^{\prime}$

Solving the simultaneous equations gives $\mathrm{v}_{\mathrm{f}}=\mathrm{v}_{0} / 2$ and $\mathrm{v}^{\prime}=-\mathrm{v}_{0} / 2$ (the block moves to the left)

## 1985M1

a. $y=y_{0}+v_{0 y} t+1 / 2 a t^{2}$ where $v_{0 y}=(50 \mathrm{~m} / \mathrm{s}) \sin 37^{\circ}=30 \mathrm{~m} / \mathrm{s}$
$0=(35 \mathrm{~m})+(30 \mathrm{~m} / \mathrm{s}) \mathrm{t}-1 / 2\left(10 \mathrm{~m} / \mathrm{s}^{2}\right) \mathrm{t}^{2}$
$\mathrm{t}=7 \mathrm{~s}$
b. $\quad R=v_{x} t=(50 \mathrm{~m} / \mathrm{s}) \cos 37^{\circ}(7 \mathrm{~s})=280 \mathrm{~m}$
c. $\mathrm{v}_{\mathrm{A}}=\mathrm{v}_{\mathrm{x}}=40 \mathrm{~m} / \mathrm{s}$
$\mathrm{v}_{\mathrm{B}}=\mathrm{v}_{0}=50 \mathrm{~m} / \mathrm{s}$
$1 / 2 \mathrm{mv}_{\mathrm{C}}^{2}=1 / 2 \mathrm{mv}_{0}^{2}+\mathrm{mgh}$ gives $\mathrm{v}_{\mathrm{C}}=56.6 \mathrm{~m} / \mathrm{s}$
d. The center of mass of the pieces will land at the same location as the intact projectile. Putting the center of mass at the same location gives $\mathrm{x}(10 \mathrm{~kg})=(30 \mathrm{~m})(6 \mathrm{~kg})$ and $\mathrm{x}=18 \mathrm{~m}$

## 1991M1

a. $\quad \mathrm{p}_{\mathrm{i}}=\mathrm{p}_{\mathrm{f}}$
$\mathrm{mv}_{0}=3 \mathrm{mv}$
$\mathrm{v}=\mathrm{v}_{0} / 3$
b. Conservation of energy: $\mathrm{K}_{\mathrm{i}}=\mathrm{K}_{\mathrm{f}}+\mathrm{U}_{\mathrm{f}}$
$1 / 2(3 \mathrm{~m})\left(\mathrm{v}_{0} / 3\right)^{2}=3 \mathrm{mgr}+\mathrm{K}_{\mathrm{f}}$
$\mathrm{K}_{\mathrm{f}}=\mathrm{mv}_{0}{ }^{2} / 6-3 \mathrm{mgr}$
c. The minimum speed needed to execute a complete loop at the top of the loop is found form dynamics:
$\Sigma \mathrm{F}=\mathrm{ma}_{\mathrm{c}}$
$\mathrm{F}_{\mathrm{N}}+\mathrm{mg}=\mathrm{mv}^{2} / \mathrm{r}$, the minimum speed is found when $\mathrm{F}_{\mathrm{N}}=0$
$\mathrm{mg}=\mathrm{mv}_{\text {top }}{ }^{2} / \mathrm{r}$ giving $\mathrm{v}_{\text {top }}=(\mathrm{gr})^{1 / 2}$
When the bullet is moving at that minimum speed, the speed after the collision is $\mathrm{v}_{\text {min }} / 3$ as above
Now use conservation of energy to find the speed needed by the bullet and block after the collision
$\mathrm{K}_{\mathrm{i}}=\mathrm{K}_{\mathrm{f}}+\mathrm{U}_{\mathrm{f}}$
$1 / 2(3 \mathrm{~m})\left(\mathrm{v}_{\text {min }} / 3\right)^{2}=3 \mathrm{mg}(2 \mathrm{r})+1 / 2(3 \mathrm{~m})\left(\mathrm{v}_{\text {top }}\right)^{2}$ giving $\mathrm{v}_{\text {min }}=3(5 \mathrm{gr})^{1 / 2}$

## 1991M3

a. $\quad \mathrm{F}=\mathrm{kD}=\mathrm{mg}$
$\mathrm{k}=\mathrm{mg} / \mathrm{D}$
b. When $\mathrm{v}_{\text {relative }}=0$ the separation is neither increasing nor decreasing. This is the transition from approaching vs separating, where the spring is about to start expanding. Additionally, when they are moving the same speed, the kinetic energy is at a minimum (totally inelastic collision) therefore $U_{s}$ is at a maximum.
c. $p_{i}=p_{f} ; \mathrm{mv}_{0}=3 \mathrm{mv}^{\prime}$ and $\mathrm{K}_{\mathrm{i}}=\mathrm{K}_{\mathrm{f}}+\mathrm{U}_{\mathrm{s}} ; 1 / 2 \mathrm{mv}_{0}{ }^{2}=1 / 2 \mathrm{kx}^{2}+1 / 2(3 \mathrm{~m}) \mathrm{v}^{\prime 2}$

Using $\mathrm{v}^{\prime}=\mathrm{v}_{0} / 3$ and $\mathrm{k}=\mathrm{mg} / \mathrm{D}$ yields $\mathrm{x}=\mathrm{v}_{0}(2 \mathrm{D} / 3 \mathrm{~g})^{1 / 2}$
d. This is an elastic collision: $\mathrm{mv}_{0}=\mathrm{mv}_{\mathrm{I}}+2 \mathrm{mv}_{\text {II }}$ and $\mathrm{v}_{0}=\mathrm{v}_{\mathrm{II}}-\mathrm{v}_{\mathrm{I}}$

Solving the simultaneous equations gives $\mathrm{v}_{\mathrm{II}}=(2 / 3) \mathrm{v}_{0}$

## 1992M1

a. $\mathrm{U}=\mathrm{K} ; \mathrm{mgH}=1 / 2 \mathrm{mv}^{2}$ gives $\mathrm{v}=10 \mathrm{~m} / \mathrm{s}$
b. $h_{b}=v_{0} t-1 / 2 \mathrm{gt}^{2}$ and $h_{c}=H-1 / 2 \mathrm{gt}^{2}$

They will collide when $h_{b}=h_{c}$ which gives $v_{0} t=H$, or $t=H / v_{0}=(5 \mathrm{~m}) /(10 \mathrm{~m} / \mathrm{s})=0.5 \mathrm{~s}$
c. Substituting $t$ into the height expressions gives $h_{c}=3.8 \mathrm{~m}$
d. $\mathrm{v}_{\mathrm{b}}=\mathrm{v}_{0}-\mathrm{gt}=5 \mathrm{~m} / \mathrm{s}$
$v_{c}=g t=5 \mathrm{~m} / \mathrm{s}$
e. $\quad m_{b} v_{b}-m_{c} v_{c}=\left(m_{b}+m_{c}\right) v^{\prime}$ giving $v^{\prime}=4 \mathrm{~m} / \mathrm{s}$ (up)

## 1993M1

a. $\quad \mathrm{U}=1 / 2 \mathrm{kx}^{2}=50 \mathrm{~J}$
b. $\quad U+W_{f}=K_{f}$
$50 \mathrm{~J}-\mu \mathrm{m}_{\mathrm{c}} \mathrm{gd}=1 / 2 \mathrm{~m}_{\mathrm{c}} \mathrm{v}_{\mathrm{c}}{ }^{2}$ giving $\mathrm{v}_{\mathrm{c}}=4.58 \mathrm{~m} / \mathrm{s}$
c. $\quad m_{c} v_{c}=\left(m_{c}+m_{D}\right) v_{f}$ gives $v_{f}=3.05 \mathrm{~m} / \mathrm{s}$
d. When the blocks come to rest $K_{i}+W_{f}=0$, or $1 / 2\left(m_{c}+m_{D}\right) v_{f}^{2}=\mu\left(m_{c}+m_{D}\right) g d$

Solving gives $\mathrm{d}-1.16 \mathrm{~m}$

## 1994M1

a. $\quad \mathrm{U}=1 / 2 \mathrm{kx}^{2}=16 \mathrm{~J}$
b. $\quad 1 / 2 \mathrm{~m}_{\mathrm{tot}} \mathrm{v}^{2}=\mathrm{U} ; \mathrm{v}=(2 \mathrm{U} / \mathrm{m})^{1 / 2}=3.9 \mathrm{~m} / \mathrm{s}$
c. $\quad \mathrm{m}_{\mathrm{c}} \mathrm{v}_{\mathrm{i}}=\mathrm{m}_{\text {tot }} \mathrm{V}$
$\mathrm{v}_{\mathrm{i}}=\mathrm{m}_{\text {tot }} \mathrm{v} / \mathrm{m}_{\mathrm{c}}=81.9 \mathrm{~m} / \mathrm{s}$
d. Less. The kinetic energy of the center of mass of the system in this case is non-zero, so some of the initial energy remains as kinetic energy, so the spring compression is less.
e. This is an elastic collision: $\mathrm{Mv}=\mathrm{Mv}_{1}+\mathrm{M}_{8} \mathrm{v}_{2}$ and $\mathrm{v}=\mathrm{v}_{2}-\mathrm{v}_{1}$
$3.9 \mathrm{~m} / \mathrm{s}=\mathrm{v}_{2}-\mathrm{v}_{1}$ and $(2.1 \mathrm{~kg})(3.9 \mathrm{~m} / \mathrm{s})=(2.1 \mathrm{~kg}) \mathrm{v}_{1}+(8 \mathrm{~kg}) \mathrm{v}_{2}$

## 1995M1

a. Impulse is the area under the curve $=12 \mathrm{~N}-\mathrm{s}$
b. $\quad \mathrm{J}=\Delta \mathrm{p} ; 12 \mathrm{~N}-\mathrm{s}=\mathrm{m}_{\mathrm{b}} \Delta \mathrm{v}_{\mathrm{b}}=(5 \mathrm{~kg})\left(\mathrm{v}_{\mathrm{fb}}-0\right)$
$\mathrm{v}_{\mathrm{fb}}=2.4 \mathrm{~m} / \mathrm{s}$
c. i. $J=\Delta \mathrm{p} ;-12 \mathrm{~N}-\mathrm{s}=\mathrm{m}_{\mathrm{c}} \Delta \mathrm{v}_{\mathrm{c}}=(0.5 \mathrm{~kg})\left(\mathrm{v}_{\mathrm{fc}}-26 \mathrm{~m} / \mathrm{s}\right)$
$\mathrm{v}_{\mathrm{fc}}=2 \mathrm{~m} / \mathrm{s}$
ii. Since $\mathrm{v}_{\mathrm{fc}}>0$ the cube is moving to the right
d. $\quad \Delta \mathrm{K}=\mathrm{K}_{\mathrm{f}}-\mathrm{K}_{\mathrm{i}}=1 / 2\left(\mathrm{~m}_{\mathrm{c}} \mathrm{v}_{\mathrm{fc}}{ }^{2}+\mathrm{m}_{\mathrm{b}} \mathrm{v}_{\mathrm{fb}}{ }^{2}-\mathrm{m}_{\mathrm{c}} \mathrm{v}_{\mathrm{ic}}{ }^{2}\right)$
$|\Delta K|=154 \mathrm{~J}$
e. Both objects take the same time to reach the floor: $t=(2 y / g)^{1 / 2}=0.5 \mathrm{~s}$
$\mathrm{x}=\mathrm{vt} ; \mathrm{x}_{\mathrm{b}}=\mathrm{v}_{\mathrm{fb}} \mathrm{t}=1.2 \mathrm{~m}$ and $\mathrm{x}_{\mathrm{c}}=\mathrm{v}_{\mathrm{fc}} \mathrm{t}=1 \mathrm{~m}$
The separation is $\Delta \mathrm{x}=\mathrm{x}_{\mathrm{b}}-\mathrm{x}_{\mathrm{c}}=0.2 \mathrm{~m}$

## 1997M2

a. The total mass is the sum of the car's mass plus the mass of sand that has fallen up to that time
$\mathrm{M}=\mathrm{M}_{0}+(\Delta \mathrm{M} / \Delta \mathrm{t}) \mathrm{t}=\mathrm{M}_{0}+\mathrm{Ct}$
b. $p_{i}=p_{f}$
$\mathrm{M}_{0} \mathrm{v}_{0}=\mathrm{Mv}^{\prime}=\left(\mathrm{M}_{0}+\mathrm{Ct}\right) \mathrm{v}^{\prime}$
$\mathrm{v}^{\prime}=\mathrm{M}_{0} \mathrm{v}_{0} /\left(\mathrm{M}_{0}+\mathrm{Ct}\right)$
c. i. $\mathrm{K}_{\mathrm{i}}=1 / 2 \mathrm{M}_{0} \mathrm{v}_{0}{ }^{2}$
ii. $\mathrm{K}_{\mathrm{f}}=1 / 2 \operatorname{Mv}(\mathrm{~T})^{2}=1 / 2\left(\mathrm{M}_{0}+\mathrm{CT}\right)\left(\mathrm{M}_{0} \mathrm{v}_{0} /\left(\mathrm{M}_{0}+\mathrm{CT}\right)\right)^{2}=1 / 2 \mathrm{M}_{0} \mathrm{v}_{0}{ }^{2}\left(\mathrm{M}_{0} /\left(\mathrm{M}_{0}+\mathrm{CT}\right)\right)$
iii. KE is not conserved as it is dissipated in the inelastic collisions of the sand and car. Also, since momentum is constant and mass increases, $\mathrm{K}=\mathrm{p}^{2} / 2 \mathrm{~m}$ will decrease
d. i. $\mathrm{F}_{\mathrm{N}}=\mathrm{M}_{0} \mathrm{~g}$
ii. $\mathrm{F}_{\mathrm{N}}=$ the weight of the cart plus the weight of the sand accumulated plus the impulsive force required to stop the vertical motion of the sand: $\mathrm{F}_{\mathrm{N}}=\mathrm{M}_{0} \mathrm{~g}+\mathrm{M}_{\mathrm{s}} \mathrm{g}+\mathrm{F}_{\mathrm{I}}$
$\mathrm{M}_{\mathrm{s}}=\mathrm{Ct}$ and $\mathrm{F}_{\mathrm{I}}=\Delta \mathrm{p} / \Delta \mathrm{t}=(\Delta \mathrm{M} / \Delta \mathrm{t}) \mathrm{v}_{\mathrm{y}}$ where $\mathrm{v}_{\mathrm{y}}=(2 \mathrm{gh})^{1 / 2}$
$\mathrm{F}_{\mathrm{N}}=\left(\mathrm{M}_{0}+\mathrm{Ct}\right) \mathrm{g}+\mathrm{C}(2 \mathrm{gh})^{1 / 2}$
iii. $\mathrm{F}_{\mathrm{N}}=\left(\mathrm{M}_{0}+\mathrm{CT}\right) \mathrm{g}$

## 1998M1

a. $\quad \mathrm{V}_{\text {avg }}=\Delta \mathrm{x} / \Delta \mathrm{t}$
i. $\mathrm{v}_{\text {avg }}=(0.3 \mathrm{~m}-0.1 \mathrm{~m}) /(0.3 \mathrm{~s}-0.1 \mathrm{~s})=1 \mathrm{~m} / \mathrm{s}$
ii. $v_{\text {avg }}=(0.99 \mathrm{~m}-0.87 \mathrm{~m}) /(1.1 \mathrm{~s}-0.9 \mathrm{~s})=0.6 \mathrm{~m} / \mathrm{s}$
iii. $\mathrm{v}_{\text {avg }}=(1.18 \mathrm{~m}-1.14 \mathrm{~m}) /(1.9 \mathrm{~s}-1.7 \mathrm{~s})=0.2 \mathrm{~m} / \mathrm{s}$
b.

c. $\mathrm{m}_{\mathrm{A}} \mathrm{v}_{\mathrm{A}}=\mathrm{m}_{\mathrm{A}} \mathrm{v}_{\mathrm{Af}}+\mathrm{m}_{\mathrm{B}} \mathrm{V}_{\mathrm{Bf}}$ and $\mathrm{v}_{\mathrm{A}}=\mathrm{v}_{\mathrm{Bf}}-\mathrm{v}_{\mathrm{Af}}$ (elastic head on collision) gives $\mathrm{v}_{\mathrm{B}}=1.2 \mathrm{~m} / \mathrm{s}$
d.

e. i. Yes, the collision is elastic

The spring's force is conservative so the energy stored is the energy released also $K_{f}=K_{i}$ ii. At $\mathrm{t}=1 \mathrm{~s}$, the spring stores the lost KE

## 1999M1

a. conservation of momentum: $\mathrm{mv}_{0}=\left(\mathrm{m}+\mathrm{M}_{0}\right)$ v gives $\mathrm{v}=\mathrm{mv}_{0} /\left(\mathrm{m}+\mathrm{M}_{0}\right)$ conservation of energy: $1 / 2 M_{\text {total }} v^{2}=M_{\text {total }}$ gh where $h=\boldsymbol{\ell}(1-\cos \theta)$
Substituting for $\mathrm{M}_{\text {total }}$ and v gives $\mathrm{v}_{0}=\left(\mathrm{m}+\mathrm{M}_{0}\right)(2 \mathrm{~g} \ell(1-\cos \theta))^{1 / 2} / \mathrm{m}$
b. $\quad \Sigma \mathrm{F}=\mathrm{ma}$
$\mathrm{T}-\mathrm{M}_{\text {total }} \mathrm{g}=\mathrm{M}_{\text {total }} \mathrm{v}^{2} / \ell$
$T=\left(m+M_{0}\right) g+\left(m+M_{0}\right)\left(\mathrm{mv}_{0} /\left(m+M_{0}\right)\right)^{2} / \ell=\left(m+M_{0}\right) g(3-2 \cos \theta)$
c. Points were awarded for the following:

A practical procedure that uses some or all of the apparatus listed and would work
Recognition of any assumptions that must be made
Indication of the proper mathematical computation using the variables measured
d. While traditional drag force methods will work here, here is an alternate solution:
$\int F d t=\Delta p$
$\int_{0}^{\infty}-b v d t=-m v_{0}$
$\mathrm{v}=\mathrm{dx} / \mathrm{dt}$ so we can replace v dt with dx also noting that as t reaches $\propto, \mathrm{x}=\mathrm{L}$
$\int_{0}^{L}-b d x=-m v_{0}$
$-b L=-m v_{0}$
$L=m v_{0} / b$

## $\underline{2001 \mathrm{M} 1}$

a. $\quad a_{\text {avg }}=\Delta v / \Delta t=(-0.18 \mathrm{~m} / \mathrm{s}-0.22 \mathrm{~m} / \mathrm{s}) /(0.37 \mathrm{~s}-0.33 \mathrm{~s})=-10 \mathrm{~m} / \mathrm{s}^{2}$
b. $\quad \Delta \mathrm{p}=$ area under the curve in the second graph $=0.6 \mathrm{~N}-\mathrm{s}$
c. $\Delta \mathrm{p}=\mathrm{m} \Delta \mathrm{v}$
$\mathrm{m}=\Delta \mathrm{p} / \Delta \mathrm{v}=(0.6 \mathrm{~N}-\mathrm{s}) /(0.4 \mathrm{~m} / \mathrm{s})=1.5 \mathrm{~kg}$
d. $\quad|\Delta \mathrm{E}|=\left|\mathrm{E}_{\mathrm{f}}-\mathrm{E}_{\mathrm{i}}\right|=\left|1 / 2 \mathrm{mv}_{\mathrm{f}}^{2}-1 / 2 \mathrm{mv}_{\mathrm{i}}^{2}\right|=0.012 \mathrm{~J}$

## 2004M1

a. $\quad \mathrm{U}=\mathrm{K} ; \mathrm{m}_{1} \mathrm{gL}=1 / 2 \mathrm{~m}_{1} \mathrm{v}_{\mathrm{B}}{ }^{2} ; \mathrm{v}_{\mathrm{B}}=(2 \mathrm{gL})^{1 / 2}$
b. $\quad \Sigma F=m a$
$\mathrm{T}-\mathrm{m}_{1} \mathrm{~g}=\mathrm{m}_{1} \mathrm{v}_{\mathrm{B}}{ }^{2} / \mathrm{r}$
$\mathrm{T}=\mathrm{m}_{1} \mathrm{~g}+\mathrm{m}_{1}(2 \mathrm{gL}) / \mathrm{L}=3 \mathrm{~m}_{1} \mathrm{~g}$
c. $\quad \mathrm{m}_{1} \mathrm{v}_{\mathrm{B}}=\left(\mathrm{m}_{1}+\mathrm{m}_{2}\right) \mathrm{v}_{\text {after }}$
$\mathrm{v}_{\mathrm{after}}=\mathrm{m}_{1} \mathrm{v}_{\mathrm{B}} /\left(\mathrm{m}_{1}+\mathrm{m}_{2}\right)=\mathrm{m}_{1}(2 \mathrm{gL})^{1 / 2} /\left(\mathrm{m}_{1}+\mathrm{m}_{2}\right)$
d. $\quad \mathrm{K}_{\text {before }}=\mathrm{U}_{\text {before }}=\mathrm{m}_{1} \mathrm{gL}$
$K_{\text {after }}=1 / 2\left(m_{1}+m_{2}\right) v_{\text {after }}{ }^{2}=m_{1}{ }^{2} g L /\left(m_{1}+m_{2}\right)$
$\mathrm{K}_{\mathrm{b}} / \mathrm{K}_{\mathrm{a}}=\left(\mathrm{m}_{1}+\mathrm{m}_{2}\right) / \mathrm{m}_{1}$
e. To fall to the water: $y=1 / 2 \mathrm{gt}^{2}=\mathrm{L}$ so $\mathrm{t}=(2 \mathrm{~L} / \mathrm{g})^{1 / 2}$

From B to D, $\mathrm{x}_{\mathrm{BD}}=\mathrm{v}_{\text {after }} \mathrm{t}=\left(\mathrm{m}_{1}(2 \mathrm{gL})^{1 / 2} /\left(\mathrm{m}_{1}+\mathrm{m}_{2}\right)\right)(2 \mathrm{~L} / \mathrm{g})^{1 / 2}=2 \mathrm{~m}_{1} \mathrm{~L} /\left(\mathrm{m}_{1}+\mathrm{m}_{2}\right)$
From A to D: $\mathrm{x}_{\text {total }}=\mathrm{x}_{\mathrm{BD}}+\mathrm{L}=\left(3 \mathrm{~m}_{1}+\mathrm{m}_{2}\right) \mathrm{L} /\left(\mathrm{m}_{1}+\mathrm{m}_{2}\right)$

## 2010M3

This is also from the work-energy chapter. Only part d is new here
a.

$$
v=\int a d t=\int_{0}^{t} a_{\max } \sin \frac{\pi t}{T} d t=-\left.\frac{a_{\max } T}{\pi} \cos \frac{\pi t}{T}\right|_{0} ^{t}=\frac{a_{\max } T}{\pi}\left(1-\cos \frac{\pi t}{T}\right)
$$

b. $\quad W=\Delta K=1 / 2 m\left(v_{f}{ }^{2}-v_{i}{ }^{2}\right)$
$\mathrm{v}_{\mathrm{f}}=\mathrm{v}(\mathrm{T})=2 \mathrm{a}_{\text {max }} \mathrm{T} / \pi$
$\mathrm{v}_{\mathrm{i}}=\mathrm{v}(0)=0$
$\mathrm{W}=2 \mathrm{ma}_{\max }{ }^{2} \mathrm{~T}^{2} / \pi^{2}$
c. $\quad \Sigma \mathrm{F}=\mathrm{F}_{\text {rope }}-\mathrm{mg} \sin \theta=\mathrm{ma}$ where $\mathrm{a}=0$ at terminal velocity
$\mathrm{F}_{\text {rope }}=\mathrm{mg} \sin \theta$
d.

$$
J=\int F d t=\int_{0}^{T} m a_{\max } \sin \frac{\pi t}{T} d t=-\left.\frac{m a_{\max } T}{\pi} \cos \frac{\pi t}{T}\right|_{0} ^{T}=\frac{2 m a_{\max } T}{\pi}
$$

## Chapter 5

## Rotation



## SECTION A - Torque and Statics

1. Torque is the rotational analogue of
(A) kinetic energy
(B) linear momentum
(C) acceleration
(D) force
(E) mass

2. A square piece of plywood on a horizontal tabletop is subjected to the two horizontal forces shown above. Where should a third force of magnitude 5 newtons be applied to put the piece of plywood into equilibrium?
(A)

(B)

(C)

(D)

(E)


3. A uniform rigid bar of weight W is supported in a horizontal orientation as shown above by a rope that makes a $30^{\circ}$ angle with the horizontal. The force exerted on the bar at point O , where it is pivoted, is best represented by a vector whose direction is which of the following?
(A)

(B)
 $(\mathrm{C}) \longrightarrow$ (D)

(E)


4. A rod of negligible mass is pivoted at a point that is off-center, so that length $1_{1}$ is different from length $1_{2}$. The figures above show two cases in which masses are suspended from the ends of the rod. In each case the unknown mass $m$ is balanced by a known mass, $M_{1}$ or $M_{2}$, so that the rod remains horizontal. What is the value of $m$ in terms of the known masses?
(A) $\mathrm{M}_{1}+\mathrm{M}_{2}$
(B) $1 / 2\left(\mathrm{M}_{1}+\mathrm{M}_{2}\right)$
(C) $\mathrm{M}_{1} \mathrm{M}_{2}$
(D) $1 / 2 \mathrm{M}_{1} \mathrm{M}_{2}$
(E) $\sqrt{M_{1} M_{2}}$

5. A system of two wheels fixed to each other is free to rotate about a frictionless axis through the common center of the wheels and perpendicular to the page. Four forces are exerted tangentially to the rims of the wheels, as shown above. The magnitude of the net torque on the system about the axis is
(A) zero
(B) FR
(C) 2 FR
(D) 5 FR
(E) 14 FR

6. For the wheel-and-axle system shown above, which of the following expresses the condition required for the system to be in static equilibrium?
(A) $\mathrm{m}_{1}=\mathrm{m}_{2}$
(B) $\mathrm{am}_{1}=\mathrm{bm}_{2}$
(C) $\mathrm{am}_{2}=\mathrm{bm}_{1}$
(D) $a^{2} m_{l}=b^{2} m^{2}$
(E) $b^{2} m_{1}=a^{2} m_{2}$

## SECTION B - Rotational Kinematics and Dynamics

## Questions 1-2

A cylinder rotates with constant angular acceleration about a fixed axis. The cylinder's moment of inertia about the axis is $4 \mathrm{~kg} \mathrm{~m}^{2}$. At time $\mathrm{t}=0$ the cylinder is at rest. At time $\mathrm{t}=2$ seconds its angular velocity is 1 radian per second.

1. What is the angular acceleration of the cylinder between $t=0$ and $t=2$ seconds?
(A) $0.5 \mathrm{radian} / \mathrm{s}^{2}$
(B) $1 \mathrm{radian} / \mathrm{s}^{2}$
(C) $2 \mathrm{radian} / \mathrm{s}^{2}$
(D) 4 radian $/ \mathrm{s}^{2}$
(E) $5 \mathrm{radian} / \mathrm{s}^{2}$
2. What is the kinetic energy of the cylinder at time $t=2$ seconds?
(A) 1 J
(B) 2 J
(C) 3 J
(D) 4 J
(E) cannot be determined without knowing the radius of the cylinder
3. A particle is moving in a circle of radius 2 meters according to the relation $\theta=3 \mathrm{t}^{2}+2 \mathrm{t}$, where $\theta$ is measured in radians and $t$ in seconds. The speed of the particle at $t=4$ seconds is
(A) $13 \mathrm{~m} / \mathrm{s}$
(B) $16 \mathrm{~m} / \mathrm{s}$
(C) $26 \mathrm{~m} / \mathrm{s}$
(D) $52 \mathrm{~m} / \mathrm{s}$
(E) $338 \mathrm{~m} / \mathrm{s}$
4. A uniform stick has length $L$. The moment of inertia about the center of the stick is $I_{0}$. A particle of mass $M$ is attached to one end of the stick. The moment of inertia of the combined system about the center of the stick is
(A) $I_{0}+\frac{1}{4} M L^{2}$
(B) $I_{0}+\frac{1}{2} M L^{2}$
(C) $I_{0}+\frac{3}{4} M L^{2}$
(D) $I_{0}+M L^{2}$
(E) $I_{0}+\frac{5}{4} M L^{2}$

5. A light rigid rod with masses attached to its ends is pivoted about a horizontal axis as shown above. When released from rest in a horizontal orientation, the rod begins to rotate with an angular acceleration of magnitude
(A) $\frac{g}{7 l}$
(B) $\frac{g}{5 l}$
(C) $\frac{g}{4 l}$
(D) $\frac{5 g}{7 l}$
(E) $\frac{g}{l}$

6. In which of the following diagrams is the torque about point O equal in magnitude to the torque about point X in the diagram above? (All forces lie in the plane of the paper.)
(A)

(B)

(C)

(D)


## (E) None of the above

## Questions 7-8



An ant of mass $m$ clings to the rim of a flywheel of radius $r$, as shown above. The flywheel rotates clockwise on a horizontal shaft $S$ with constant angular velocity $\omega$. As the wheel rotates, the ant revolves past the stationary points I, II, III, and IV. The ant can adhere to the wheel with a force much greater than its own weight.
7. It will be most difficult for the ant to adhere to the wheel as it revolves past which of the four points?
(A) I
(B) II
(C) III
(D) IV
(E) It will be equally difficult for the ant to adhere to the wheel at all points.
8. What is the magnitude of the minimum adhesion force necessary for the ant to stay on the flywheel at point III?
(A) mg
(B) $m \omega^{2} r$
(C) $m \omega^{2} r^{2}+m g$
(D) $m \omega^{2} r-m g$
(E) $m \omega^{2} r+m g$
9. A turntable that is initially at rest is set in motion with a constant angular acceleration $\alpha$. What is the angular velocity of the turntable after it has made one complete revolution?
(A) $\sqrt{2 \alpha}$
(B) $\sqrt{2 \pi \alpha}$
(C) $\sqrt{4 \pi \alpha}$
(D) $2 \alpha$
(E) $4 \pi \alpha$

10. A 5 -kilogram sphere is connected to a 10-kilogram sphere by a rigid rod of negligible mass, as shown above. The sphere-rod combination can be pivoted about an axis that is perpendicular to the plane of the page and that passes through one of the five lettered points. Through which point should the axis pass for the moment of inertia of the sphere-rod combination about this axis to be greatest?
(A) A
(B) B
(C) C
(D) D
(E) E

## Questions 11-12

A wheel with rotational inertia $I$ is mounted on a fixed, frictionless axle. The angular speed $\omega$ of the wheel is increased from zero to $\omega_{f}$ in a time interval T.
11. What is the average net torque on the wheel during this time interval?
(A) $\frac{\omega_{f}}{T}$
(B) $\frac{\omega_{f}}{T^{2}}$
(C) $\frac{I \omega_{f}^{2}}{T}$
(D) $\frac{I \omega_{f}}{T^{2}}$
(E) $\frac{I \omega_{f}}{T}$
12. What is the average power input to the wheel during this time interval?
(A) $\frac{I \omega_{f}}{2 T}$
(B) $\frac{I \omega_{f}^{2}}{2 T}$
(C) $\frac{I \omega_{f}^{2}}{2 T^{2}}$
(D) $\frac{I^{2} \omega_{f}}{2 T^{2}}$
(E) $\frac{I^{2} \omega_{f}^{2}}{2 T^{2}}$

13. Two blocks are joined by a light string that passes over the pulley shown above, which has radius $R$ and moment of inertia $I$ about its center. $T_{l}$ and $T_{2}$ are the tensions in the string on either side of the pulley and $\alpha$ is the angular acceleration of the pulley. Which of the following equations best describes the pulley's rotational motion during the time the blocks accelerate?
(A) $m_{2} g R=I \alpha$
(B) $\left(T_{1}+T_{2}\right) R=I \alpha$
(C) $T_{2} R=I \alpha$
(D) $\left(T_{2}-T_{1}\right) R=I \alpha$
(E) $\left(m_{2}-m_{I}\right) g R=I \alpha$
14. a disk is free to rotate about an axis perpendicular to the disk through its center. If the disk starts from rest and accelerates uniformly at the rate of 3 radians $/ \mathrm{s}^{2}$ for 4 s , its angular displacement during this time is
(A) 6 radians
(B) 12 radians
(C) 18 radians
(D) 24 radians
(E) 48 radians

Questions 15-16


A solid cylinder of mass $m$ and radius $R$ has a string wound around it. A person holding the string pulls it vertically upward, as shown above, such that the cylinder is suspended in midair for a brief time interval $\Delta t$ and its center of mass does not move. The tension in the string is $T$, and the rotational inertia of the cylinder about its axis is $\frac{1}{2} M R^{2}$
15. the net force on the cylinder during the time interval $\Delta t$ is
(A) $T$
(B) $m g$
(C) $T-m g R$
(D) $m g R-T$
(E) zero
16. The linear acceleration of the person's hand during the time interval $\Delta t$ is
(A) $\frac{T-m g}{m}$
(B) $2 g$
(C) $\frac{g}{2}$
(D) $\frac{T}{m}$
(E) zero

17. A block of mass $m$ is placed against the inner wall of a hollow cylinder of radius $R$ that rotates about a vertical axis with a constant angular velocity $\omega$, as shown above. In order for friction to prevent the mass from sliding down the wall, the coefficient of static friction $\mu$ between the mass and the wall must satisfy which of the following inequalities?
(A) $\mu \geq m g$
(B) $\mu \geq \frac{g}{\omega^{2} R}$
(C) $\mu \geq \frac{\omega^{2} R}{g}$
(D) $\mu \leq \frac{g}{\omega^{2} R}$
(E) $\mu \leq \frac{\omega^{2} R}{g}$

## SECTION C - Rolling

1. A bowling ball of mass $M$ and radius $R$. whose moment of inertia about its center is $(2 / 5) M R^{2}$, rolls without slipping along a level surface at speed $v$. The maximum vertical height to which it can roll if it ascends an incline is
(A) $\frac{v^{2}}{5 g}$
(B) $\frac{2 v^{2}}{5 g}$
(C) $\frac{v^{2}}{2 g}$
(D) $\frac{7 v^{2}}{10 g}$
(E) $\frac{v^{2}}{g}$
2. A wheel of mass $M$ and radius $R$ rolls on a level surface without slipping. If the angular velocity of the wheel is $\omega$, what is its linear momentum?
(A) $\mathrm{M} \omega \mathrm{R}$
(B) $M \omega^{2} R$
(C) $\mathrm{M} \omega \mathrm{R}^{2}$
(D) $\mathrm{M} \omega^{2} \mathrm{R}^{2} / 2$
(E) Zero

Questions 3-4


A sphere of mass $M$, radius $r$, and rotational inertia $I$ is released from rest at the top of an inclined plane of height $h$ as shown above.
3. If the plane is frictionless, what is the speed $v_{c m}$, of the center of mass of the sphere at the bottom of the incline?
(A) $\sqrt{2 g h}$
(B) $\frac{2 M g h}{I}$
(C) $\frac{2 M g h r^{2}}{I}$
(D) $\sqrt{\frac{2 M g h r^{2}}{I}}$
(E) $\sqrt{\frac{2 M g h r^{2}}{I+M r^{2}}}$
4. If the plane has friction so that the sphere rolls without slipping, what is the speed $v_{c m}$ of the center of mass at the bottom of the incline?
(A) $\sqrt{2 g h}$
(B) $\frac{2 M g h}{I}$
(C) $\frac{2 M g h r^{2}}{I}$
(D) $\sqrt{\frac{2 M g h r^{2}}{I}}$
(E) $\sqrt{\frac{2 M g h r^{2}}{I+M r^{2}}}$
5. A wheel of 0.5 m radius rolls without slipping on a horizontal surface. The axle of the wheel advances at constant velocity, moving a distance of 20 m in 5 s . The angular speed of the wheel about its point of contact on the surface is
(A) 2 radians $\cdot \mathrm{s}^{-1}$
(B) 4 radians $\cdot \mathrm{s}^{-1}$
(C) 8 radians $\cdot \mathrm{s}^{-1}$
(D) 16 radians $\cdot \mathrm{s}^{-1}$
(E) 32 radians $\cdot \mathrm{s}^{-1}$
6. A car travels forward with constant velocity. It goes over a small stone, which gets stuck in the groove of a tire. The initial acceleration of the stone, as it leaves the surface of the road, is
(A) vertically upward
(B) horizontally forward
(C) horizontally backward
(D) zero
(E) upward and forward, at approximately $45^{\circ}$ to the horizontal

## SECTION D - Angular Momentum

1. An ice skater is spinning about a vertical axis with arms fully extended. If the arms are pulled in closer to the body, in which of the following ways are the angular momentum and kinetic energy of the skater affected?

|  | Angular Momentum |  |
| :--- | :--- | :--- |
|  |  | Kinetic Energy |
| (A) Increases |  | Increases |
| (B) Increases |  | Remains Constant |
| (C) Remains Constant |  | Increases |
| (D) Remains Constant |  | Remains Constant |
| (E) Decreases |  | Remains Constant |

2. A cylinder rotates with constant angular acceleration about a fixed axis. The cylinder's moment of inertia about the axis is $4 \mathrm{~kg} \mathrm{~m}^{2}$. At time $\mathrm{t}=0$ the cylinder is at rest. At time $\mathrm{t}=2$ seconds its angular velocity is 1 radian per second. What is the angular momentum of the cylinder at time $t=2$ seconds?
(A) $1 \mathrm{~kg} \mathrm{~m}^{2} / \mathrm{s}$
(B) $2 \mathrm{~kg} \mathrm{~m}^{2} / \mathrm{s}$
(C) $3 \mathrm{~kg} \mathrm{~m}^{2} / \mathrm{s}$
(D) $4 \mathrm{~kg} \mathrm{~m}^{2} / \mathrm{s}$
(E) It cannot be determined without knowing the radius of the cylinder.
3. A figure skater is spinning on frictionless ice with her arms fully extended horizontally. She then drops her arms to her sides. Which of the following correctly describes her rotational kinetic energy and angular momentum as her arms fall?

Rotational Kinetic
Energy
(A) Remains constant
(B) Decreases
(C) Decreases
(D) Increases
(E) Increases

Angular
Momentum
Remains constant
Increases
Decreases
Decreases
Remains constant

4. A particle of mass $m$ moves with a constant speed $v$ along the dashed line $y=a$. When the $x$-coordinate of the particle is $x_{0}$, the magnitude of the angular momentum of the particle with respect to the origin of the system is
(A) zero
(B) $m v a$
(C) $m v x_{o}$
(D) $m v \sqrt{x_{0}{ }^{2}+a^{2}}$
(E) $\frac{m v a}{\sqrt{x_{0}{ }^{2}+a^{2}}}$

5. The rigid body shown in the diagram above consists of a vertical support post and two horizontal crossbars with spheres attached. The masses of the spheres and the lengths of the crossbars are indicated in the diagram. The body rotates about a vertical axis along the support post with constant angular speed $\omega$. If the masses of the support post and the crossbars are negligible, what is the ratio of the angular momentum of the two upper spheres to that of the two lower spheres?
(A) $2 / 1$
(B) $1 / 1$
(C) $1 / 2$
(D) $1 / 4$
(E) $1 / 8$

6. A long board is free to slide on a sheet of frictionless ice. As shown in the top view above, a skater skates to the board and hops onto one end, causing the board to slide and rotate. In this situation, which of the following occurs?
(A) Linear momentum is converted to angular momentum.
(B) Kinetic energy is converted to angular momentum.
(C) Rotational kinetic energy is conserved.
(D) Translational kinetic energy is conserved.
(E) Linear momentum and angular momentum are both conserved.

7. A disk sliding on a horizontal surface that has negligible friction collides with a rod that is free to move and rotate on the surface, as shown in the top view above. Which of the following quantities must be the same for the disk-rod system before and after the collision?
I. Linear momentum
II. Angular momentum
III. Kinetic energy
(A) I only
(B) II only
(C) I and II only
(D) II and III only
(E) I, II, and III
8. A figure skater goes into a spin with arms fully extended. Which of the following describes the changes in the rotational kinetic energy and angular momentum of the skater as the skater's arms are brought toward the body?

Rotational

Kinetic Energy
(A) Remains the same
(B) Remains the same
(C) Increases
(D) Decreases
(E) Decreases

Angular Momentum
Increases
Remains the same
Remains the same
Increases
Remains the same

## SECTION A - Torque and Statics



2008 M2. The horizontal uniform rod shown above has length 0.60 m and mass 2.0 kg . The left end of the rod is attached to a vertical support by a frictionless hinge that allows the rod to swing up or down. The right end of the rod is supported by a cord that makes an angle of $30^{\circ}$ with the rod. A spring scale of negligible mass measures the tension in the cord. A 0.50 kg block is also attached to the right end of the rod.
a. On the diagram below, draw and label vectors to represent all the forces acting on the rod. Show each force vector originating at its point of application.
b. Calculate the reading on the spring scale.

The rotational inertia of a rod about its center is $\frac{1}{12} M L^{2}$, where $M$ is the mass of the rod and $L$ is its length.
c. Calculate the rotational inertia of the rod-block system about the hinge.
d. If the cord that supports the rod is cut near the end of the rod, calculate the initial angular acceleration of the rod-block system about the hinge.

## SECTION B - Rotational Kinematics and Dynamics



1973M3. A ball of mass $m$ is attached by two strings to a vertical rod. as shown above. The entire system rotates at constant angular velocity $\omega$ about the axis of the rod.
a. Assuming $\omega$ is large enough to keep both strings taut, find the force each string exerts on the ball in terms of $\omega$, $\mathrm{m}, \mathrm{g}, \mathrm{R}$, and $\theta$.
b. Find the minimum angular velocity, $\omega_{\text {min }}$ for which the lower string barely remains taut.


1976M2. A cloth tape is wound around the outside of a uniform solid cylinder (mass M, radius R) and fastened to the ceiling as shown in the diagram above. The cylinder is held with the tape vertical and then released from rest. As the cylinder descends, it unwinds from the tape without slipping. The moment of inertia of a uniform solid cylinder about its center is $1 / 2 \mathrm{MR}^{2}$.

a. On the circle above draw vectors showing all the forces acting on the cylinder after it is released. Label each force clearly.
b. In terms of g , find the downward acceleration of the center of the cylinder as it unrolls from the tape.
c. While descending, does the center of the cylinder move toward the left, toward the right, or straight down? Explain.


E:gure:


Eigure II

1978M1. An amusement park ride consists of a ring of radius A from which hang ropes of length $\ell$ with seats for the riders as shown in Figure I. When the ring is rotating at a constant angular velocity $\omega$ each rope forms a constant angle $\theta$ with the vertical as shown in Figure II. Let the mass of each rider be $m$ and neglect friction, air resistance, and the mass of the ring, ropes, and seats.
a. In the space below, draw and label all the forces acting on rider X (represented by the point below) under the constant rotating condition of Figure II. Clearly define any symbols you introduce.
b. Derive an expression for $\omega$ in terms of $\mathrm{A}, \ell, \theta$ and the acceleration of gravity $g$.
c. Determine the minimum work that the motor that powers the ride would have to perform to bring the system from rest to the constant rotating condition of Figure II. Express your answer in terms of $\mathrm{m}, \mathrm{g}, \boldsymbol{\ell}, \theta$, and the speed $v$ of each rider.


1983M2. A uniform solid cylinder of mass $\mathrm{m}_{1}$ and radius $R$ is mounted on frictionless bearings about a fixed axis through $O$. The moment of inertia of the cylinder about the axis is $I=1 / 2 m_{1} R^{2}$. A block of mass $m_{2}$, suspended by a cord wrapped around the cylinder as shown above, is released at time $t=0$.
a. On the diagram below draw and identify all of the forces acting on the cylinder and on the block.

b. In terms of $\mathrm{m}_{1}, \mathrm{~m}_{2}, R$. and g , determine each of the following.
i. The acceleration of the block
ii. The tension in the cord


1985M3. A pulley of mass 3 m and radius r is mounted on frictionless bearings and supported by a stand of mass 4 m at rest on a table as shown above. The moment of inertia of this pulley about its axis is $1.5 \mathrm{mr}^{2}$. Passing over the pulley is a massless cord supporting a block of mass m on the left and a block of mass 2 m on the right. The cord does not slip on the pulley, so after the block-pulley system is released from rest, the pulley begins to rotate.
a. On the diagrams below, draw and label all the forces acting on each block.

b. Use the symbols identified in part a. to write each of the following.
i. The equations of translational motion (Newton's second law) for each of the two blocks
ii. The analogous equation for the rotational motion of the pulley
c. Solve the equations in part b. for the acceleration of the two blocks.
d. Determine the tension in the segment of the cord attached to the block of mass $m$.
e. Determine the normal force exerted on the apparatus by the table while the blocks are in motion.


1988M3. The two uniform disks shown above have equal mass, and each can rotate on frictionless bearings about a fixed axis through its center. The smaller disk has a radius R and moment of inertia I about its axis. The larger disk has a radius 2 R
a. Determine the moment of inertia of the larger disk about its axis in terms of I.

The two disks are then linked as shown below by a light chain that cannot slip. They are at rest when, at time $t=$ 0 , a student applies a torque to the smaller disk, and it rotates counterclockwise with constant angular acceleration $\alpha$. Assume that the mass of the chain and the tension in the lower part of the chain, are negligible. In terms of I, R, $\alpha$, and t , determine each of the following:

b. The angular acceleration of the larger disk
c. The tension in the upper part of the chain
d. The torque that the student applied to the smaller disk
e. The rotational kinetic energy of the smaller disk as a function of time


1989M2. Block A of mass 2M hangs from a cord that passes over a pulley and is connected to block B of mass 3M that is free to move on a frictionless horizontal surface, as shown above. The pulley is a disk with frictionless bearings, having a radius $R$ and moment of inertia $3 M R R^{2}$. Block $C$ of mass $4 M$ is on top of block $B$. The surface between blocks B and C is NOT frictionless. Shortly after the system is released from rest, block A moves with a downward acceleration a, and the two blocks on the table move relative to each other.

In terms of $\mathrm{M}, \mathrm{g}$, and a , determine the
a. tension $\mathrm{T}_{\mathrm{v}}$ in the vertical section of the cord
b. tension $\mathrm{T}_{\mathrm{h}}$ in the horizontal section of the cord

If $\mathrm{a}=2$ meters per second squared, determine the
c. coefficient of kinetic friction between blocks B and C
d. acceleration of block C


1991M2. Two masses. $\mathrm{m}_{1}$ and $\mathrm{m}_{2}$ are connected by light cables to the perimeters of two cylinders of radii $\mathrm{r}_{1}$ and $\mathrm{r}_{2}$, respectively. as shown in the diagram above. The cylinders are rigidly connected to each other but are free to rotate without friction on a common axle. The moment of inertia of the pair of cylinders is $\mathrm{I}=45 \mathrm{~kg} \bullet \mathrm{~m}^{2}$ Also $\mathrm{r}_{1}=0.5$ meter, $\mathrm{r}_{2}=1.5$ meters, and $\mathrm{m}_{1}=20$ kilograms.
a. Determine $m_{2}$ such that the system will remain in equilibrium.

The mass $m_{2}$ is removed and the system is released from rest.
b. Determine the angular acceleration of the cylinders.
c. Determine the tension in the cable supporting $\mathrm{m}_{1}$
d. Determine the linear speed of $m_{1}$ at the time it has descended 1.0 meter.


1993M3. A long, uniform rod of mass $M$ and length $\mathbb{A}$ s supported at the left end by a horizontal axis into the page and perpendicular to the rod, as shown above. The right end is connected to the ceiling by a thin vertical thread so that the rod is horizontal. The moment of inertia of the rod about the axis at the end of the $\operatorname{rod}$ is $M l^{2} / 3$. Express the answers to all parts of this question in terms of $M, \boldsymbol{\ell}$, and $g$.
a. Determine the magnitude and direction of the force exerted on the rod by the axis.

The thread is then burned by a match. For the time immediately after the thread breaks, determine each of the following:
b. The angular acceleration of the rod about the axis
c. The translational acceleration of the center of mass of the rod
d. The force exerted on the end of the rod by the axis

The rod rotates about the axis and swings down from the horizontal position.
e. Determine the angular velocity of the rod as a function of $\theta$, the arbitrary angle through which the rod has swung.


1999M3 As shown above, a uniform disk is mounted to an axle and is free to rotate without friction. A thin uniform rod is rigidly attached to the disk so that it will rotate with the disk. A block is attached to the end of the rod. Properties of the disk, rod, and block are as follows.

Disk: $\quad$ mass $=3 \mathrm{~m}$, radius $=R$, moment of inertia about center $I_{D}=1.5 \mathrm{mR}^{2}$
Rod: mass $=m$, length $=2 R$, moment of inertia about one end $I_{R}=4 / 3\left(\mathrm{mR}^{2}\right)$
Block: mass $=2 \mathrm{~m}$

The system is held in equilibrium with the rod at an angle $\theta_{0}$ to the vertical, as shown above, by a horizontal string of negligible mass with one end attached to the disk and the other to a wall. Express your answers to the following in terms of $\mathrm{m}, \mathrm{R}, \theta_{0}$, and g .
a. Determine the tension in the string.

The string is now cut, and the disk-rod-block system is free to rotate.
b. Determine the following for the instant immediately after the string is cut.
i. The magnitude of the angular acceleration of the disk
ii. The magnitude of the linear acceleration of the mass at the end of the rod


As the disk rotates, the rod passes the horizontal position shown above.
c. Determine the linear speed of the mass at the end of the rod for the instant the rod is in the horizontal position.


2000M3. A pulley of radius $R_{1}$ and rotational inertia $I_{1}$ is mounted on an axle with negligible friction. A light cord passing over the pulley has two blocks of mass $m$ attached to either end, as shown above. Assume that the cord does not slip on the pulley. Determine the answers to parts a . and b . in terms of $m, R_{1}, I_{1}$, and fundamental constants.
a. Determine the tension T in the cord.
b. One block is now removed from the right and hung on the left. When the system is released from rest, the three blocks on the left accelerate downward with an acceleration $\mathrm{g} / 3$. Determine the following.
i. The tension $\mathrm{T}_{3}$ in the section of cord supporting the three blocks on the left
ii. The tension $\mathrm{T}_{1}$ in the section of cord supporting the single block on the right
iii. The rotational inertia $I_{1}$ of the pulley

c. The blocks are now removed and the cord is tied into a loop, which is passed around the original pulley and a second pulley of radius $2 \mathrm{R}_{1}$ and rotational inertia $16 \mathrm{I}_{1}$. The axis of the original pulley is attached to a motor that rotates it at angular speed $\omega_{1}$, which in turn causes the larger pulley to rotate. The loop does not slip on the pulleys. Determine the following in terms of $\mathrm{I}_{1}, \mathrm{R}_{\mathrm{I}}$, and $\omega_{1}$.
i. The angular speed $\omega_{2}$ of the larger pulley
ii. The angular momentum $L_{2}$ of the larger pulley
iii. The total kinetic energy of the system


## Experiment A

2001M3. A light string that is attached to a large block of mass $4 m$ passes over a pulley with negligible rotational inertia and is wrapped around a vertical pole of radius r, as shown in Experiment A above. The system is released from rest, and as the block descends the string unwinds and the vertical pole with its attached apparatus rotates. The apparatus consists of a horizontal rod of length 2 L , with a small block of mass $m$ attached at each end. The rotational inertia of the pole and the rod are negligible.
a. Determine the rotational inertia of the rod-and-block apparatus attached to the top of the pole.
b. Determine the downward acceleration of the large block.
c. When the large block has descended a distance $D$, how does the instantaneous total kinetic energy of the three blocks compare with the value $4 m g D$ ? Check the appropriate space below and justify your answer.

Greater than 4 mgD $\qquad$ Equal to $4 \mathrm{mgD} \ldots \quad$ Less than 4 mgD $\qquad$


Experiment B
The system is now reset. The string is rewound around the pole to bring the large block back to its original location. The small blocks are detached from the rod and then suspended from each end of the rod, using strings of length $l$. The system is again released from rest so that as the large block descends and the apparatus rotates, the small blocks swing outward, as shown in Experiment B above. This time the downward acceleration of the block decreases with time after the system is released.
d. When the large block has descended a distance $D$, how does the instantaneous total kinetic energy of the three blocks compare to that in part c.? Check the appropriate space below and justify your answer.

Greater before $\qquad$ Equal to before $\qquad$ Less than before $\qquad$


2003M3. Some physics students build a catapult, as shown above. The supporting platform is fixed firmly to the ground. The projectile, of mass 10 kg , is placed in cup $A$ at one end of the rotating arm. A counterweight bucket $B$ that is to be loaded with various masses greater than 10 kg is located at the other end of the arm. The arm is released from the horizontal position, shown in Figure 1, and begins rotating. There is a mechanism (not shown) that stops the arm in the vertical position, allowing the projectile to be launched with a horizontal velocity as shown in Figure 2.
a. The students load five different masses in the counterweight bucket, release the catapult, and measure the resulting distance x traveled by the 10 kg projectile, recording the following data.

| Mass (kg) | 100 | 300 | 500 | 700 | 900 |
| ---: | :---: | :---: | :---: | :---: | :---: |
| $x(\mathrm{~m})$ | 18 | 37 | 45 | 48 | 51 |

i. The data are plotted on the axes below. Sketch a best-fit curve for these data points.

ii. Using your best-fit curve, determine the distance $x$ traveled by the projectile if 250 kg is placed in the counterweight bucket.
b. The students assume that the mass of the rotating arm, the cup, and the counterweight bucket can be neglected. With this assumption, they develop a theoretical model for $x$ as a function of the counterweight mass using the relationship $x=v_{x} t$, where $v$, is the horizontal velocity of the projectile as it leaves the cup and $t$ is the time after launch.
i. How many seconds after leaving the cup will the projectile strike the ground?
ii. Derive the equation that describes the gravitational potential energy of the system relative to the ground when in the position shown in Figure 1, assuming the mass in the counterweight bucket is $M$.
iii. Derive the equation for the velocity of the projectile as it leaves the cup, as shown in Figure 2.
c. i. Complete the theoretical model by writing the relationship for $x$ as a function of the counterweight mass using the results from b. i and b. iii.
ii. Compare the experimental and theoretical values of $x$ for a counterweight bucket mass of 300 kg . Offer a reason for any difference.


2004M2. A solid disk of unknown mass and known radius $R$ is used as a pulley in a lab experiment, as shown above. A small block of mass $m$ is attached to a string, the other end of which is attached to the pulley and wrapped around it several times. The block of mass $m$ is released from rest and takes a time $t$ to fall the distance $D$ to the floor.
a. Calculate the linear acceleration $a$ of the falling block in terms of the given quantities.
b. The time $t$ is measured for various heights $D$ and the data are recorded in the following table.

| $D(\mathrm{~m})$ | $t(\mathrm{~s})$ |
| :---: | :---: |
| 0.5 | 0.68 |
| 1 | 1.02 |
| 1.5 | 1.19 |
| 2 | 1.38 |

i. What quantities should be graphed in order to best determine the acceleration of the block? Explain your reasoning.
ii. On the grid below, plot the quantities determined in b. i., label the axes, and draw the best-fit line to the data.

iii. Use your graph to calculate the magnitude of the acceleration.
c. Calculate the rotational inertia of the pulley in terms of $m, R, a$, and fundamental constants.
d. The value of acceleration found in b.iii, along with numerical values for the given quantities and your answer to c., can be used to determine the rotational inertia of the pulley. The pulley is removed from its support and its rotational inertia is found to be greater than this value. Give one explanation for this discrepancy.

## SECTION C - Rolling



1974M2. The moment of inertia of a uniform solid sphere (mass M, radius R ) about a diameter is $2 \mathrm{MR}^{2} / 5$. The sphere is placed on an inclined plane (angle $\theta$ ) as shown above and released from rest.
a. Determine the minimum coefficient of friction $\mu$ between the sphere and plane with which the sphere will roll down the incline without slipping
b. If $\mu$ were zero, would the speed of the sphere at the bottom be greater, smaller, or the same as in part a.? Explain your answer.

1977M2. A uniform cylinder of mass M, and radius $R$ is initially at rest on a rough horizontal surface. The moment of inertia of a cylinder about its axis is $1 / 2 \mathrm{MR}^{2}$. A string, which is wrapped around the cylinder, is pulled upwards with a force T whose magnitude is 0.6 Mg and whose direction is maintained vertically upward at all times. In consequence, the cylinder both accelerates horizontally and slips. The coefficient of kinetic friction is 0.5 .
a. On the diagram below, draw vectors that represent each of the forces acting on the cylinder identify and clearly label each force.

b. Determine the linear acceleration a of the center of the cylinder.
c. Calculate the angular acceleration $\alpha$ of the cylinder.
d. Your results should show that a and $\alpha \mathrm{R}$ are not equal. Explain.


1980M3. A billiard ball has mass M, radius $R$, and moment of inertia about the center of mass $I_{c}=2 \mathrm{MR}^{2} / 5$ The ball is struck by a cue stick along a horizontal line through the ball's center of mass so that the ball initially slides with a velocity $\mathrm{v}_{\mathrm{o}}$ as shown above. As the ball moves across the rough billiard table (coefficient of sliding friction $\mu_{\mathrm{k}}$ ), its motion gradually changes from pure translation through rolling with slipping to rolling without slipping.
a. Develop an expression for the linear velocity v of the center of the ball as a function of time while it is rolling with slipping.
b. Develop an expression for the angular velocity $\omega$ of the ball as a function of time while it is rolling with slipping.
c. Determine the time at which the ball begins to roll without slipping.
d. When the ball is struck it acquires an angular momentum about the fixed point P on the surface of the table. During the subsequent motion the angular momentum about point P remains constant despite the frictional force. Explain why this is so.


1986M2. An inclined plane makes an angle of $\theta$ with the horizontal, as shown above. A solid sphere of radius R and mass $M$ is initially at rest in the position shown, such that the lowest point of the sphere is a vertical height $h$ above the base of the plane. The sphere is released and rolls down the plane without slipping. The moment of inertia of the sphere about an axis through its center is $2 \mathrm{MR}^{2} / 5$. Express your answers in terms of $\mathrm{M}, \mathrm{R} . h, \mathrm{~g}$, and $\theta$.
a. Determine the following for the sphere when it is at the bottom of the plane:
i. Its translational kinetic energy
ii. Its rotational kinetic energy
b. Determine the following for the sphere when it is on the plane.
i. Its linear acceleration
ii. The magnitude of the frictional force acting on it

The solid sphere is replaced by a hollow sphere of identical radius R and mass M . The hollow sphere, which is released from the same location as the solid sphere, rolls down the incline without slipping.
c. What is the total kinetic energy of the hollow sphere at the bottom of the plane?
d. State whether the rotational kinetic energy of the hollow sphere is greater than, less than, or equal to that of the solid sphere at the bottom of the plane. Justify your answer.


1990M2. A block of mass $m$ slides up the incline shown above with an initial speed $v_{O}$ in the position shown.
a. If the incline is frictionless, determine the maximum height H to which the block will rise, in terms of the given quantities and appropriate constants.
b. If the incline is rough with coefficient of sliding friction $\mu$, determine the maximum height to which the block will rise in terms of H and the given quantities.


A thin hoop of mass $m$ and radius $R$ moves up the incline shown above with an initial speed $v_{O}$ in the position shown.
c. If the incline is rough and the hoop rolls up the incline without slipping, determine the maximum height to which the hoop will rise in terms of H and the given quantities.
d. If the incline is frictionless, determine the maximum height to which the hoop will rise in terms of H and the given quantities.


Note: Diagram not drawn to scale.
1994M2. A large sphere rolls without slipping across a horizontal surface. The sphere has a constant translational speed of 10 meters per second, a mass $m$ of 25 kilograms, and a radius $r$ of 0.2 meter. The moment of inertia of the sphere about its center of mass is $I=2 \mathrm{mr}^{2} / 5$. The sphere approaches a $25^{\circ}$ incline of height 3 meters as shown above and rolls up the incline without slipping.
a. Calculate the total kinetic energy of the sphere as it rolls along the horizontal surface.
b. i. Calculate the magnitude of the sphere's velocity just as it leaves the top of the incline. ii. Specify the direction of the sphere's velocity just as it leaves the top of the incline.
c. Neglecting air resistance, calculate the horizontal distance from the point where the sphere leaves the incline to the point where the sphere strikes the level surface.
d. Suppose, instead, that the sphere were to roll toward the incline as stated above, but the incline were frictionless. State whether the speed of the sphere just as it leaves the top of the incline would be less than, equal to, or greater than the speed calculated in b. Explain briefly.


1997M3. A solid cylinder with mass M, radius R, and rotational inertia $1 / 2 \mathrm{MR}^{2}$ rolls without slipping down the inclined plane shown above. The cylinder starts from rest at a height H . The inclined plane makes an angle $\theta$ with the horizontal. Express all solutions in terms of M, R, H, $\theta$, and g.
a. Determine the translational speed of the cylinder when it reaches the bottom of the inclined plane.
b. On the figure below, draw and label the forces acting on the cylinder as it rolls down the inclined plane Your arrow should begin at the point of application of each force.

c. Show that the acceleration of the center of mass of the cylinder while it is rolling down the inclined plane is $(2 / 3) g \sin \theta$.
d. Determine the minimum coefficient of friction between the cylinder and the inclined plane that is required for the cylinder to roll without slipping.
e. The coefficient of friction $\mu$ is now made less than the value determined in part d., so that the cylinder both rotates and slips.
i. Indicate whether the translational speed of the cylinder at the bottom of the inclined plane is greater than, less than, or equal to the translational speed calculated in part a. Justify your answer.
ii. Indicate whether the total kinetic energy of the cylinder at the bottom of the inclined plane is greater than, less than, or equal to the total kinetic energy for the previous case of rolling without slipping. Justify your answer.


2002M2. The cart shown above is made of a block of mass $m$ and four solid rubber tires each of mass $\mathrm{m} / 4$ and radius $r$. Each tire may be considered to be a disk. (A disk has rotational inertia $1 / 2 M L^{2}$, where M is the mass and $L$ is the radius of the disk.) The cart is released from rest and rolls without slipping from the top of an inclined plane of height $h$. Express all algebraic answers in terms of the given quantities and fundamental constants.
a. Determine the total rotational inertia of all four tires.
b. Determine the speed of the cart when it reaches the bottom of the incline.
c. After rolling down the incline and across the horizontal surface, the cart collides with a bumper of negligible mass attached to an ideal spring, which has a spring constant k . Determine the distance $\mathrm{x}_{\mathrm{m}}$ the spring is compressed before the cart and bumper come to rest.
d. Now assume that the bumper has a non-negligible mass. After the collision with the bumper, the spring is compressed to a maximum distance of about $90 \%$ of the value of $x_{m}$ in part $c$.. Give a reasonable explanation for this decrease.


2006M3. A thin hoop of mass $M$, radius $R$, and rotational inertia $M R^{2}$ is released from rest from the top of the ramp of length $L$ above. The ramp makes an angle $\theta$ with respect to a horizontal tabletop to which the ramp is fixed. The table is a height $H$ above the floor. Assume that the hoop rolls without slipping down the ramp and across the table. Express all algebraic answers in terms of given quantities and fundamental constants.
a. Derive an expression for the acceleration of the center of mass of the hoop as it rolls down the ramp.
b. Derive an expression for the speed of the center of mass of the hoop when it reaches the bottom of the ramp.
c. Derive an expression for the horizontal distance from the edge of the table to where the hoop lands on the floor.
d. Suppose that the hoop is now replaced by a disk having the same mass $M$ and radius $R$. How will the distance from the edge of the table to where the disk lands on the floor compare with the distance determined in part c . for the hoop?

Less than $\qquad$ The same as $\qquad$ Greater than $\qquad$
Briefly justify your response.


Note: Figure not drawn to scale.
2010M2. A bowling ball of mass 6.0 kg is released from rest from the top of a slanted roof that is 4.0 m long and angled at $30^{\circ}$, as shown above. The ball rolls along the roof without slipping. The rotational inertia of a sphere of mass $M$ and radius $R$ about its center of mass is $2 \mathrm{MR}^{2} / 5$.
a. On the figure below, draw and label the forces (not components) acting on the ball at their points of application as it rolls along the roof.

b. Calculate the force due to friction acting on the ball as it rolls along the roof. If you need to draw anything other than what you have shown in part a. to assist in your solution, use the space below. Do NOT add anything to the figure in part a.
c. Calculate the linear speed of the center of mass of the ball when it reaches the bottom edge of the roof.
d. A wagon containing a box is at rest on the ground below the roof so that the ball falls a vertical distance of 3.0 m and lands and sticks in the center of the box. The total mass of the wagon and the box is 12 kg . Calculate the horizontal speed of the wagon immediately after the ball lands in it.

## SECTION D - Angular Momentum



1975M2. A bicycle wheel of mass $M$ (assumed to be concentrated at its rim) and radius $R$ is mounted horizontally so it may turn without friction on a vertical axle. A dart of mass $m_{o}$ is thrown with velocity $v_{o}$ as shown above and sticks in the tire.
a. If the wheel is initially at rest, find its angular velocity $\omega$ after the dart strikes.
b. In terms of the given quantities, determine the ratio:
final kinetic energy of the system
initial kinetic energy of the system


Betore Coilisicn


1978M2. A system consists of a mass $M_{2}$ and a uniform rod of mass $M_{1}$ and length $l$. The rod is initially rotating with an angular speed $\omega$ on a horizontal frictionless table about a vertical axis fixed at one end through point $P$. The moment of inertia of the rod about P is $\mathrm{M}^{2} / 3$. The rod strikes the stationary mass $\mathrm{M}_{2}$. As a result of this collision, the rod is stopped and the mass $\mathrm{M}_{2}$ moves away with speed v .
a. Using angular momentum conservation determine the speed v in terms of $\mathrm{M}_{1}, \mathrm{M}_{2}, l$, and $\omega$.
b. Determine the linear momentum of this system just before the collision in terms of $\mathbf{M}_{1}, l$, and $\omega$.
c. Determine the linear momentum of this system just after the collision in terms of $\mathrm{M}_{1} l$, and $\omega$.
d. What is responsible for the change in the linear momentum of this system during the collision?
e. Why is the angular momentum of this system about point P conserved during the collision?


Figure I: Before
1981M3. A thin, uniform rod of mass $M_{1}$ and length $L$, is initially at rest on a frictionless horizontal surface. The moment of inertia of the rod about its center of mass is $\mathrm{M}_{1} \mathrm{~L}^{2} / 12$. As shown in Figure I , the rod is struck at point P by a mass $\mathrm{m}_{2}$ whose initial velocity v is perpendicular to the rod. After the collision, mass $\mathrm{m}_{2}$ has velocity
$-1 / 2 \mathbf{v}$ as shown in Figure II. Answer the following in terms of the symbols given.
a. Using the principle of conservation of linear momentum, determine the velocity $\mathbf{v}^{\prime}$ of the center of mass of this rod after the collision.
b. Using the principle of conservation of angular momentum, determine the angular velocity $\omega$ of the rod about its center of mass after the collision.
c. Determine the change in kinetic energy of the system resulting from the collision.


1982M3. A system consists of two small disks, of masses m and 2 m , attached to a rod of negligible mass of length $3 l$ as shown above. The rod is free to turn about a vertical axis through point P . The two disks rest on a rough horizontal surface; the coefficient of friction between the disks and the surface is $\mu$. At time $t=0$, the rod has an initial counterclockwise angular velocity $\omega_{0}$ about $P$. The system is gradually brought to rest by friction.
Develop expressions for the following quantities in terms of $\mu \mathrm{m}, l, \mathrm{~g}$, and $\omega_{\mathrm{o}}$
a. The initial angular momentum of the system about the axis through $P$
b. The frictional torque acting on the system about the axis through $P$
c. The time T at which the system will come to rest.


Note: You may use $g=10 \mathrm{~m} / \mathrm{s}^{2}$.
1987M3. A 1.0-kilogram object is moving horizontally with a velocity of 10 meters per second, as shown above, when it makes a glancing collision with the lower end of a bar that was hanging vertically at rest before the collision. For the system consisting of the object and bar, linear momentum is not conserved in this collision, but kinetic energy is conserved. The bar, which has a length $l$ of 1.2 meters and a mass $m$ of 3.0 kilograms, is pivoted about the upper end. Immediately after the collision the object moves with speed $v$ at an angle $\theta$ relative to its original direction. The bar swings freely, and after the collision reaches a maximum angle of $90^{\circ}$ with respect to the vertical. The moment of inertia of the bar about the pivot is $\mathrm{I}_{\mathrm{bar}}=\mathrm{m} l^{2} / 3$ Ignore all friction.
a. Determine the angular velocity of the bar immediately after the collision.
b. Determine the speed $v$ of the l-kilogram object immediately after the collision.
c. Determine the magnitude of the angular momentum of the object about the pivot just before the collision.
d. Determine the angle $\theta$.


1992M2. Two identical spheres, each of mass M and negligible radius, are fastened to opposite ends of a rod of negligible mass and length $2 l$. This system is initially at rest with the rod horizontal, as shown above, and is free to rotate about a frictionless, horizontal axis through the center of the rod and perpendicular to the plane of the page. A bug, of mass 3 M , lands gently on the sphere on the left. Assume that the size of the bug is small compared to the length of the rod. Express your answers to all parts of the question in terms of $\mathrm{M}, l$, and physical constants.
a. Determine the torque about the axis immediately after the bug lands on the sphere.
b. Determine the angular acceleration of the rod-spheres-bug system immediately after the bug lands.


The rod-spheres-bug system swings about the axis. At the instant that the rod is vertical, as shown above, determine each of the following.
c. The angular speed of the bug
d. The angular momentum of the system
e. The magnitude and direction of the force that must be exerted on the bug by the sphere to keep the bug from being thrown off the sphere


1996M3. Consider a thin uniform rod of mass M and length $l$, as shown above.
a. Show that the rotational inertia of the rod about an axis through its center and perpendicular to its length is $\mathrm{M} l^{2} / 12$.


The rod is now glued to a thin hoop of mass M and radius $\mathrm{R} / 2$ to form a rigid assembly, as shown above. The centers of the rod and the hoop coincide at point P . The assembly is mounted on a horizontal axle through point P and perpendicular to the page.
b. What is the rotational inertia of the rod-hoop assembly about the axle?

Several turns of string are wrapped tightly around the circumference of the hoop. The system is at rest when a cat, also of mass M , grabs the free end of the string and hangs vertically from it without swinging as it unwinds, causing the rod-hoop assembly to rotate. Neglect friction and the mass of the string.
c. Determine the tension T in the string.
d. Determine the angular acceleration a of the rod-hoop assembly.
e. Determine the linear acceleration of the cat.
f. After descending a distance $\mathrm{H}=5 l / 3$, the cat lets go of the string. At that instant, what is the angular momentum of the cat about point P ?

1998M2. A space shuttle astronaut in a circular orbit around the Earth has an assembly consisting of two small dense spheres, each of mass m , whose centers are connected by a rigid rod of length $l$ and negligible mass. The astronaut also has a device that will launch a small lump of clay of mass $m$ at speed $v_{0}$. Express your answers in terms of $\mathrm{m}, \mathrm{v}_{0} l$. and fundamental constants.

a. Initially, the assembly is "floating" freely at rest relative to the cabin, and the astronaut launches the clay lump so that it perpendicularly strikes and sticks to the midpoint of the rod, as shown above.
i. Determine the total kinetic energy of the system (assembly and clay lump) after the collision.
ii. Determine the change in kinetic energy as a result of the collision.

b. The assembly is brought to rest, the clay lump removed, and the experiment is repeated as shown above, with the clay lump striking perpendicular to the rod but this time sticking to one of the spheres of the assembly.
i. Determine the distance from the left end of the rod to the center of mass of the system (assembly and clay lump) immediately after the collision. (Assume that the radii of the spheres and clay lump are much smaller than the separation of the spheres.)
ii. On the figure above, indicate the direction of the motion of the center of mass immediately after the collision.
iii. Determine the speed of the center of mass immediately after the collision.
iv. Determine the angular speed of the system (assembly and clay lump) immediately after the collision.
v. Determine the change in kinetic energy as a result of the collision.


2005M3. A system consists of a ball of mass $M_{2}$ and a uniform rod of mass $M_{1}$ and length $d$. The rod is attached to a horizontal frictionless table by a pivot at point $P$ and initially rotates at an angular speed $\omega$, as shown above left. The rotational inertia of the rod about point P is $\frac{1}{3} M_{1} d^{2}$. The rod strikes the ball, which is initially at rest. As a result of this collision, the rod is stopped and the ball moves in the direction shown above right. Express all answers in terms of $M_{1}, M_{2}, \omega, d$, and fundamental constants.
a. Derive an expression for the angular momentum of the rod about point $P$ before the collision.
b. Derive an expression for the speed $v$ of the ball after the collision.
c. Assuming that this collision is elastic, calculate the numerical value of the ratio $M_{1} / M_{2}$

d. A new ball with the same mass $M_{l}$ as the rod is now placed a distance $x$ from the pivot, as shown above. Again assuming the collision is elastic, for what value of $x$ will the rod stop moving after hitting the ball?

## SECTION A - Torque and Statics

## Solution

Answer

1. Definition of Torque
2. To balance the forces (Fnet=0) the answer must be A or D, to prevent rotation, obviously A would be needed
3. FBD


Since the rope is at an angle it has x and y
B components of force.
Therefore, H would have to exist to counteract $\mathrm{T}_{\mathrm{x}}$. Based on $\mathrm{T}_{\text {net }}=0$ requirement, V also would have to exist to balance W if we were to chose a pivot point at the right end of the bar
4. Applying rotational equilibrium to each diagram gives

E
DIAGRAM 1: $(\mathrm{mg})\left(\mathrm{L}_{1}\right)=\left(\mathrm{M}_{1} \mathrm{~g}\right)\left(\mathrm{L}_{2}\right)$
DIAGRAM 2: $\left(\mathrm{M}_{2} \mathrm{~g}\right)\left(\mathrm{L}_{1}\right)=\operatorname{mg}\left(\mathrm{L}_{2}\right)$
$\mathrm{L}_{1}=\mathrm{M}_{1}\left(\mathrm{~L}_{2}\right) / \mathrm{m}$
(sub this $\mathrm{L}_{1}$ ) into the Diagram 2 eqn, and solve.


## SECTION B - Rotational Kinematics and Dynamics

1. $\alpha=\Delta \omega / \Delta t$

A
2. $\mathrm{K}_{\mathrm{rot}}=1 / 2 \mathrm{I} \omega^{2}$
3. $\omega=\mathrm{d} \theta / \mathrm{dt}=6 \mathrm{t}+2 ; \mathrm{v}=\omega \mathrm{r}$
4. $\mathrm{I}_{\text {tot }}=\Sigma \mathrm{I}=\mathrm{I}_{0}+\mathrm{I}_{\mathrm{M}}=\mathrm{I}_{0}+\mathrm{M}(1 / 2 \mathrm{~L})^{2}$
5. $\quad \Sigma \tau=\mathrm{I} \alpha$ where $\Sigma \tau=\left(3 \mathrm{M}_{0}\right)(\ell)-\left(\mathrm{M}_{0}\right)(2 \ell)=\mathrm{M}_{0} \ell$ and $\mathrm{I}=\left(3 \mathrm{M}_{0}\right)(\ell)^{2}+\left(\mathrm{M}_{0}\right)(2 \ell)^{2}=7 \mathrm{M}_{0} \ell^{2}$
6. $\quad \tau_{\mathrm{X}}=\mathrm{F} ; \tau_{\mathrm{O}}=\mathrm{F}_{\mathrm{O}} \mathrm{L}_{\mathrm{O}} \sin \theta$, solve for the correct combination of $\mathrm{F}_{\mathrm{O}}$ and $\mathrm{L}_{\mathrm{O}}$
5. Find the torques of each using proper signs and add up.

$$
\begin{aligned}
& +(1)-(2)+(3)+(4) \\
& +F(3 R)-(2 F)(3 R)+F(2 R)+F(3 R)=2 F R
\end{aligned}
$$


6. Simply apply rotational equilibrium

B
$\left(\mathrm{m}_{1} \mathrm{~g}\right) \cdot \mathrm{r}_{1}=\left(\mathrm{m}_{2} \mathrm{~g}\right) \cdot \mathrm{r}_{2}$
$\mathrm{m}_{1} \mathrm{a}=\mathrm{m}_{2} \mathrm{~b}$
C
7. Just as the tension in a rope is greatest at the bottom of as vertical circle, the force needed to
6. The first movement of the point of contact of a rolling object is vertically upward as there is no side to side (sliding) motion for the point in contact

## SECTION D - Angular Momentum

1. $\quad L_{i}=L_{f}$ so $I_{i} \omega_{I}=I_{f} \omega_{f}$ and since $I_{f}<I_{i}$ (mass more concentrated near axis), then $\omega_{f}>\omega_{i}$ The increase in $\omega$ is in the same proportion as the decrease in I, and the kinetic energy is proportional to $\mathrm{I} \omega^{2}$ so the increase in $\omega$ results in an overall increase in the kinetic energy. Alternately, the skater does work to pull their arms in and this work increases the KE of the skater
2. $\mathrm{L}=\mathrm{I} \omega$
3. $\quad L_{i}=L_{f}$ so $I_{i} \omega_{I}=I_{f} \omega_{f}$ and since $I_{f}<I_{i}$ (mass more concentrated near axis), then $\omega_{f}>\omega_{i}$ The increase in $\omega$ is in the same proportion as the decrease in I, and the kinetic energy is proportional to $\mathrm{I} \omega^{2}$ so the increase in $\omega$ results in an overall increase in the kinetic energy. Alternately, the skater does work to pull their arms in and this work increases the KE of the skater
4. $\quad \mathrm{L}=\operatorname{mvr}_{\perp}$ where $\mathrm{r}_{\perp}$ is the perpendicular line joining the origin and the line along which the particle is moving
5. $\quad \mathrm{L}=\mathrm{I} \omega$ and since $\omega$ is uniform the ratio $\mathrm{L}_{\text {upper }} / \mathrm{L}_{\text {lower }}=\mathrm{I}_{\text {upper }} / \mathrm{I}_{\text {lower }}=2 \mathrm{~mL}^{2} / 2(2 \mathrm{~m})(2 \mathrm{~L})^{2}=1 / 8$
6. Since it is a perfectly inelastic (sticking) collision, KE is not conserved. As there are no external forces or torques, both linear and angular momentum are conserved
7. As there are no external forces or torques, both linear and angular momentum are conserved. As the type of collision is not specified, we cannot say kinetic energy must be the same.
8. $\quad L_{i}=L_{f}$ so $I_{i} \omega_{I}=I_{f} \omega_{f}$ and since $I_{f}<I_{i}$ (mass more concentrated near axis), then $\omega_{f}>\omega_{i}$ The increase in $\omega$ is in the same proportion as the decrease in I , and the kinetic energy is proportional to $\mathrm{I} \omega^{2}$ so the increase in $\omega$ results in an overall increase in the kinetic energy. Alternately, the skater does work to pull their arms in and this work increases the KE of the skater

## SECTION A - Torque and Statics

2008M2
a.

b. $\quad \Sigma \tau=0$

About the hinge: $\mathrm{TL} \sin 30^{\circ}-\mathrm{mgL}-\mathrm{Mg}(\mathrm{L} / 2)=0$ gives $\mathrm{T}=29 \mathrm{~N}$
c. $\mathrm{I}_{\text {total }}=\mathrm{I}_{\text {rod }}+\mathrm{I}_{\text {block }}$ where $\mathrm{I}_{\text {rod,end }}=\mathrm{I}_{\mathrm{cm}}+\mathrm{MD}^{2}=\mathrm{ML}^{2} / 12+\mathrm{M}(\mathrm{L} / 2)^{2}=\mathrm{ML}^{2} / 3$
$\mathrm{I}_{\text {total }}=\mathrm{ML}^{2} / 3+\mathrm{mL}^{2}=0.42 \mathrm{~kg}-\mathrm{m}^{2}$
d. $\quad \Sigma \tau=\mathrm{I} \alpha$
$\mathrm{mgL}+\mathrm{MgL} / 2=\mathrm{I} \alpha$ gives $\alpha=21 \mathrm{rad} / \mathrm{s}^{2}$

## SECTION B - Rotational Kinematics and Dynamics

## 1973M3

a. Define a coordinate system with the x -axis directed to the vertical rod and the y -axis directed upwards and perpendicular to the first. Let $T_{1}$ be the tension in the horizontal string. Let $T_{2}$ be the tension in the string tilted upwards.
Applying Newton's Second Law: $\Sigma \mathrm{F}_{\mathrm{x}}=\mathrm{T}_{1}+\mathrm{T}_{2} \sin \theta=\mathrm{m} \omega^{2} \mathrm{R} ; \Sigma \mathrm{F}_{\mathrm{y}}=\mathrm{T}_{2} \cos \theta-\mathrm{mg}=0$
Solving yields: $\mathrm{T}_{2}=\mathrm{mg} / \cos \theta$ and $\mathrm{T}_{1}=\mathrm{m}\left(\omega^{2} \mathrm{R}-\mathrm{g} \tan \theta\right)$
b. Let $T_{1}=0$ and solving for $\omega$ gives $\omega=(g \tan \theta / R)^{1 / 2}$

## 1976M2

a.

b. $\quad \Sigma \tau=\mathrm{I} \alpha$ (about center of mass) (one could also choose about the point at which the tape comes off the cylinder)
$\mathrm{TR}=1 / 2 \mathrm{MR}^{2} \times(\mathrm{a} / \mathrm{R})$
$\mathrm{T}=1 / 2 \mathrm{Ma}$
$\Sigma \mathrm{F}=\mathrm{ma}$
$\mathrm{Mg}-\mathrm{T}=\mathrm{Ma}$
$\mathrm{Mg}=3 \mathrm{Ma} / 2$
$\mathrm{a}=2 \mathrm{~g} / 3$
c. As there are no horizontal forces, the cylinder moves straight down.

1978M1
a.

b. $\quad \Sigma \mathrm{F}=\mathrm{ma} ; \mathrm{T} \cos \theta=\mathrm{mg}$ and $\mathrm{T} \sin \theta=\mathrm{m} \omega^{2} \mathrm{r}=\mathrm{m} \omega^{2}(\mathrm{~A}+\ell \sin \theta)$
$\omega=\sqrt{\frac{g \tan \theta}{A+l \sin \theta}}$
c. $\mathrm{W}=\Delta \mathrm{E}=\Delta \mathrm{K}+\Delta \mathrm{U}=1 / 2 \mathrm{mv}^{2}+\operatorname{mg\ell }(1-\cos \theta)$ for each rider
$\mathrm{W}=6\left(1 / 2 \mathrm{mv}^{2}+\mathrm{mg} \ell(1-\cos \theta)\right)$

## 1983M2

a.

b. i./ii. On the disk: $\Sigma \tau=\mathrm{I} \alpha=\mathrm{TR}=1 / 2 \mathrm{~m}_{1} \mathrm{R}^{2} \alpha$

For the block $\mathrm{a}=\alpha \mathrm{R}$ so $\alpha=\mathrm{a} / \mathrm{R}$ and $\Sigma \mathrm{F}=\mathrm{m}_{2} \mathrm{~g}-\mathrm{T}=\mathrm{m}_{2} \mathrm{a}$
Solving yields
$a=\frac{2 m_{2} g}{2 m_{2}+m_{1}}$ and $T=\frac{m_{1} m_{2} g}{2 m_{2}+m_{1}}$
a.

b. i. $\Sigma \mathrm{F}=\mathrm{ma} ; \mathrm{T}_{1}-\mathrm{mg}=\mathrm{ma}$ and $2 \mathrm{mg}-\mathrm{T}_{2}=2 \mathrm{ma}$
ii. $\Sigma \tau=\mathrm{I} \alpha ;\left(\mathrm{T}_{2}-\mathrm{T}_{1}\right) \mathrm{r}=\mathrm{I} \alpha$
c. $\quad \alpha=a / r$

Combining equations from b.i. gives $\mathrm{T}_{2}-\mathrm{T}_{1}=\mathrm{mg}-3 \mathrm{ma}$
Substituting for $\left(\mathrm{T}_{2}-\mathrm{T}_{1}\right)$ into torque equation gives $\mathrm{a}=2 \mathrm{~g} / 9$
d. $\quad \mathrm{T}_{1}=\mathrm{m}(\mathrm{g}+\mathrm{a})=11 \mathrm{mg} / 9$
e. $\mathrm{F}_{\mathrm{N}}=7 \mathrm{mg}+\mathrm{T}_{1}+\mathrm{T}_{2}$ (the table counters all the downward forces on the apparatus)
$\mathrm{T}_{2}=2 \mathrm{~m}(\mathrm{~g}-\mathrm{a})=14 \mathrm{mg} / 9$
$\mathrm{F}_{\mathrm{N}}=88 \mathrm{mg} / 9$

## 1988M3

a. I is proportional to $\mathrm{mR}^{2}$; masses are equal and R becomes 2 R
$\mathrm{I}_{2 \mathrm{R}}=4 \mathrm{I}$
b. The disks are coupled by the chain along their rims, which means the linear motion of the rims have the same displacement, velocity and acceleration.
$\mathrm{v}_{\mathrm{R}}=\mathrm{v}_{2 \mathrm{R}} ; \mathrm{R} \omega_{\mathrm{R}}=2 \mathrm{R} \omega_{2 \mathrm{R}} ; \mathrm{R} \alpha \mathrm{t}=2 \mathrm{R} \alpha_{2 \mathrm{R}} \mathrm{t}$ gives $\alpha_{2 \mathrm{R}}=\alpha / 2$
c. $\quad \tau_{2 \mathrm{R}}=\mathrm{T}(2 \mathrm{R})=\mathrm{I}_{2 \mathrm{R}} \alpha_{2 \mathrm{R}}=(4 \mathrm{I})(\alpha / 2)=2 \mathrm{I} \alpha$ giving $\mathrm{T}=\mathrm{I} \alpha / \mathrm{R}$
d. $\quad \Sigma \tau=\tau_{\text {student }}-\mathrm{TR}=\mathrm{I} \alpha$
$\tau_{\text {student }}=\mathrm{I} \alpha+\mathrm{TR}=\mathrm{I} \alpha+(\mathrm{I} \alpha / \mathrm{R}) \mathrm{R}=2 \mathrm{I} \alpha$
e. $K=1 / 2 I \omega^{2}=1 / 2 I(\alpha t)^{2}$

## 1989M2

a. $\quad \Sigma \mathrm{F}=\mathrm{ma} ; 2 \mathrm{Mg}-\mathrm{T}_{\mathrm{v}}=2 \mathrm{Ma}$ so $\mathrm{T}_{\mathrm{v}}=2 \mathrm{M}(\mathrm{g}-\mathrm{a})$
b. $\quad \Sigma \tau=\mathrm{T}_{\mathrm{v}} \mathrm{R}-\mathrm{T}_{\mathrm{h}} \mathrm{R}=\mathrm{I} \alpha=3 \mathrm{MR}^{2}(\mathrm{a} / \mathrm{R})$
$\mathrm{T}_{\mathrm{h}}=\left(\mathrm{T}_{\mathrm{v}} \mathrm{R}-3 \mathrm{MRa}\right) / \mathrm{R}=2 \mathrm{M}(\mathrm{g}-\mathrm{a})-3 \mathrm{Ma}=2 \mathrm{Mg}-5 \mathrm{Ma}$
c. $\quad \mathrm{F}_{\mathrm{f}}=\mu \mathrm{F}_{\mathrm{N}}=\mu(4 \mathrm{Mg})$
$\mathrm{T}_{\mathrm{h}}-\mathrm{F}_{\mathrm{f}}=3 \mathrm{Ma}$
$2 \mathrm{Mg}-5 \mathrm{Ma}-4 \mu \mathrm{Mg}=3 \mathrm{Ma}$
$4 \mu \mathrm{Mg}=2 \mathrm{Mg}-8 \mathrm{Ma}$
$\mu=(2 g-8 a) / 4 g$
plugging in given values gives $\mu=0.1$
d. $\quad \mathrm{F}_{\mathrm{f}}=4 \mu \mathrm{Mg}=\mathrm{ma}_{\mathrm{C}}=4 \mathrm{Ma}_{\mathrm{C}}$
$\mathrm{a}_{\mathrm{C}}=1 \mathrm{~m} / \mathrm{s}^{2}$

## 1991M2

a. $\quad \Sigma \tau=0 ; \mathrm{m}_{2} \mathrm{gr}_{2}=\mathrm{m}_{1} \mathrm{gr}_{1} ; \mathrm{m}_{2}=\mathrm{m}_{1} \mathrm{r}_{1} / \mathrm{r}_{2}=6.67 \mathrm{~kg}$
b. $/ \mathrm{c} . \tau=\mathrm{I} \alpha ; \mathrm{Tr}_{1}=\left(45 \mathrm{~kg}-\mathrm{m}^{2}\right) \alpha$
$\Sigma \mathrm{F}=\mathrm{ma} ;(20 \mathrm{~kg}) \mathrm{g}-\mathrm{T}=(20 \mathrm{~kg}) \mathrm{a}$
Combining with $\mathrm{a}=\alpha \mathrm{r}$ gives $\alpha=2 \mathrm{rad} / \mathrm{s}^{2}$ and $\mathrm{T}=180 \mathrm{~N}$
d. $m g h=1 / 2 \mathrm{mv}^{2}+1 / 2 \mathrm{I} \omega^{2}=1 / 2 \mathrm{mv}^{2}+1 / 2 \mathrm{I}\left(\mathrm{v}^{2} / \mathrm{r}^{2}\right)$ giving $\mathrm{v}=1.4 \mathrm{~m} / \mathrm{s}$

## 1993M3

a. $\quad \Sigma \tau=\mathrm{F}_{\mathrm{a}} \ell-\mathrm{Mg} \ell / 2=0$ giving $\mathrm{F}_{\mathrm{a}}=\mathrm{Mg} / 2$
b. $\quad \Sigma \tau=\mathrm{Mg} \ell / 2=\mathrm{I} \alpha=\left(\mathrm{M}^{2} / 3\right) \alpha ; \alpha=3 \mathrm{~g} / 2 \ell$
c. $\quad \mathrm{a}=\alpha \mathrm{r}$ where $\mathrm{r}=\ell / 2$
$\mathrm{a}=(3 \mathrm{~g} / 2 \ell)(\boldsymbol{\ell} / 2)=3 \mathrm{~g} / 4$
d. $\quad \Sigma \mathrm{F}=\mathrm{Ma} ; \mathrm{Mg}-\mathrm{F}_{\mathrm{a}}=\mathrm{Ma}=\mathrm{M}(3 \mathrm{~g} / 4)$
$\mathrm{F}_{\mathrm{a}}=\mathrm{Mg} / 4$
e. $\quad \Delta U=\Delta K_{\text {rot }}$
$\operatorname{mgh}=\operatorname{mg}(\boldsymbol{\ell} / 2) \sin \theta=1 / 2 \mathrm{I} \omega^{2}=1 / 2\left(\mathrm{Ml}^{2} / 3\right) \omega^{2}$
solving gives $\omega=(3 \operatorname{gsin} \theta / \ell)^{1 / 2}$

## 1999M3

a. $\quad \Sigma \tau=0$ so $\tau_{\mathrm{cw}}=\tau_{\mathrm{ccw}}$ and $\tau_{\mathrm{cw}}=\mathrm{TR}$ (from the string) so we just need to find $\tau_{\mathrm{ccw}}$ as the sum of the torques from the various parts of the system
$\Sigma \tau_{\text {ccw }}=\tau_{\text {rod }}+\tau_{\text {block }}=\mathrm{mgR} \sin \theta_{0}+2 \mathrm{mg}(2 \mathrm{R}) \sin \theta_{0}=5 \mathrm{mgR} \sin \theta_{0}=\mathrm{TR}$ so $\mathrm{T}=5 \mathrm{mg} \sin \theta_{0}$
b. i. $\mathrm{I}=\mathrm{I}_{\text {disk }}+\mathrm{I}_{\text {rod }}+\mathrm{I}_{\text {block }}=3 \mathrm{mR}^{2} / 2+4 \mathrm{mR}^{2} / 3+2 \mathrm{~m}(2 \mathrm{R})^{2}=65 \mathrm{mR}^{2} / 6$
$\alpha=\tau / \mathrm{I}=\left(5 \mathrm{mgR} \sin \theta_{0}\right) /\left(65 \mathrm{mR}^{2} / 6\right)=6 \mathrm{~g} \sin \theta_{0} / 13 \mathrm{R}$
ii. $\mathrm{a}=\alpha \mathrm{r}$ where $\mathrm{r}=2 \mathrm{R}$ so $\mathrm{a}=12 \mathrm{~g} \sin \theta_{0} / 13$
c. $\Delta \mathrm{U}($ from each component $)=\mathrm{K}=1 / 2 \mathrm{I} \omega^{2}$
$\mathrm{mgR} \cos \theta_{0}+2 \mathrm{mg}(2 \mathrm{R}) \cos \theta_{0}=1 / 2\left(65 \mathrm{mR}^{2} / 6\right) \omega^{2}$
$\omega=\left(12 \mathrm{~g} \cos \theta_{0} / 13 \mathrm{R}\right)^{1 / 2}$ and $\mathrm{v}=\omega \mathrm{r}=\omega(2 \mathrm{R})=4\left(3 \mathrm{gR} \cos \theta_{0} / 13\right)^{1 / 2}$

## 2000M3

a.


$$
\Sigma \mathrm{F}=\mathrm{ma}=0 \text { so } \mathrm{T}=2 \mathrm{mg}
$$

b. i.

ii.

$\Sigma \mathrm{F}=\mathrm{ma}$
$\mathrm{T}_{1}-\mathrm{mg}=\mathrm{m}(\mathrm{g} / 3)$
$\mathrm{T}_{1}=4 \mathrm{mg} / 3$
iii. $\Sigma \tau=\left(\mathrm{T}_{3}-\mathrm{T}_{1}\right) \mathrm{R}_{1}=\mathrm{I} \alpha$ and $\alpha=\mathrm{a} / \mathrm{R}_{1}=\mathrm{g} / 3 \mathrm{R}_{1}$
$(2 \mathrm{mg}-4 \mathrm{mg} / 3) \mathrm{R}_{1}=\mathrm{I}_{1}\left(\mathrm{~g} / 3 \mathrm{R}_{1}\right)$
$\mathrm{I}_{1}=2 \mathrm{mR}_{1}{ }^{2}$
c. i. Tangential speeds are equal; $\omega_{1} R_{1}=\omega_{2} R_{2}=\omega_{2}\left(2 R_{1}\right)$ therefore $\omega_{2}=\omega_{1} / 2$
ii. $\mathrm{L}=\mathrm{I} \omega=\left(16 \mathrm{I}_{1}\right)\left(\omega_{1} / 2\right)=8 \mathrm{I}_{1} \omega_{1}$
iii. $K=1 / 2 I_{1} \omega_{1}{ }^{2}+1 / 2 I_{2} \omega_{2}{ }^{2}=(5 / 2) I_{1} \omega_{1}{ }^{2}$

## 2001M3

a. $\quad \mathrm{I}=\Sigma \mathrm{mr}^{2}=\mathrm{mL}^{2}+\mathrm{mL}^{2}=2 \mathrm{~mL}^{2}$
b. $\quad \Sigma \mathrm{F}=\mathrm{ma} ; 4 \mathrm{mg}-\mathrm{T}=4 \mathrm{ma}$
$\Sigma \tau=\mathrm{I} \alpha ; \mathrm{Tr}=\mathrm{I} \alpha ; \mathrm{T}=\mathrm{I} \alpha / \mathrm{r}=4 \mathrm{mg}-4 \mathrm{ma}$ and $\alpha=\mathrm{a} / \mathrm{r}$, solving gives $\mathrm{a}=2 \mathrm{gr}^{2} /\left(\mathrm{L}^{2}+2 \mathrm{r}^{2}\right)$
c. Equal, total energy is conserved
d. Less, the small blocks rise and gain potential energy. The total energy available is still 4 mgD , therefore the kinetic energy must be less than in part c .

## 2003M3

a. i.

ii. $x=33 \mathrm{~m}$
b. i. $y=1 / 2 \mathrm{gt}^{2} ; \mathrm{t}=(2 \mathrm{y} / \mathrm{g})^{1 / 2}=1.75 \mathrm{~s}$
ii. $U_{\text {initial }}=U_{\text {bucket }}+U_{\text {projectile }}=M\left(9.8 \mathrm{~m} / \mathrm{s}^{2}\right)(3 \mathrm{~m})+(10 \mathrm{~kg})\left(9.8 \mathrm{~m} / \mathrm{s}^{2}\right)(3 \mathrm{~m})=29.4 \mathrm{M}+294$
iii. $U_{\text {initial }}=U_{\text {final }}+K$ where $U_{\text {final }}=\operatorname{Mg}(1 \mathrm{~m})+(10 \mathrm{~kg}) \mathrm{g}(15 \mathrm{~m})=9.8 \mathrm{M}+1470$
$\mathrm{K}_{\text {projectile }}=1 / 210 \mathrm{v}_{\mathrm{x}}{ }^{2}$ and $\mathrm{K}_{\text {bucket }}=1 / 2 \mathrm{Mv}_{\mathrm{b}}{ }^{2}$ where $\mathrm{v}_{\mathrm{b}}=\mathrm{v}_{\mathrm{x}} / 6$
putting it all together gives $29.4 \mathrm{M}+294=9.8 \mathrm{M}+1470+5 \mathrm{v}_{\mathrm{x}}{ }^{2}+(\mathrm{M} / 72) \mathrm{v}_{\mathrm{x}}{ }^{2}$
$v_{x}=\sqrt{\frac{19.6 M-1176}{5+M / 72}}$
c. i. $x=v_{x} t$
$x=1.75 \sqrt{\frac{19.6 M-1176}{5+M / 72}}$
d. $x(300 \mathrm{~kg})=39.7 \mathrm{~m}$ (greater than the experimental value)
possible reasons include friction at the pivot, air resistance, neglected masses not negligible

## 2004M2

a. $\quad x=v_{0} t+1 / 2 a t^{2}$
$\mathrm{x}=\mathrm{D}$ and $\mathrm{v}_{0}=0$ so $\mathrm{D}=1 / 2 \mathrm{at}^{2}$ and $\mathrm{a}=2 \mathrm{D} / \mathrm{t}^{2}$
b. i. graph $D$ vs. $t^{2}$ (as an example)
ii.

iii. $\mathrm{a}=2($ slope $)=2.04 \mathrm{~m} / \mathrm{s}^{2}$
c. $\quad \Sigma \tau=\mathrm{TR}=\mathrm{I} \alpha$ and $\alpha=\mathrm{a} / \mathrm{R}$ so $\mathrm{I}=\mathrm{TR}^{2} / \mathrm{a}$
$\Sigma \mathrm{F}=\mathrm{mg}-\mathrm{T}=\mathrm{ma}$ so $\mathrm{T}=\mathrm{m}(\mathrm{g}-\mathrm{a})$
$\mathrm{I}=\mathrm{m}(\mathrm{g}-\mathrm{a}) \mathrm{R}^{2} / \mathrm{a}=\mathrm{mR}^{2}((\mathrm{~g} / \mathrm{a})-1)$
d. The string was wrapped around the pulley several times, causing the effective radius at which the torque acted to be larger than the radius of the pulley used in the calculation.

The string slipped on the pulley, allowing the block to accelerate faster than it would have otherwise, resulting in a smaller experimental moment of inertia.

Friction is not a correct answer, since the presence of friction would make the experimental value of the moment of inertia too large

## SECTION C - Rolling

NOTE: Rolling problems may be solved considering rotation about the center of mass or the point of contact. The solutions below will only show one of the two methods, but for most, if not all cases, the other method is applicable. When considering rotation about the point of contact, remember to use the parallel axis theorem for the moment of inertia of the rolling object.

## 1974M2

a. Torque provided by friction; at minimum $\mu, \mathrm{F}_{\mathrm{f}}=\mu \mathrm{F}_{\mathrm{N}}=\mu \mathrm{Mg} \cos \theta$
$\tau=\mathrm{F}_{\mathrm{f}} \mathrm{R}=\mathrm{I} \alpha=(2 / 5) \mathrm{MR}^{2}(\mathrm{a} / \mathrm{R}) ; \mathrm{F}_{\mathrm{f}}=(2 / 5) \mathrm{Ma}=\mu \mathrm{Mg} \cos \theta$ giving $\mathrm{a}=(5 / 2) \mu \mathrm{g} \cos \theta$
$\Sigma \mathrm{F}=\mathrm{Ma} ; \mathrm{Mg} \sin \theta-\mu \mathrm{Mg} \cos \theta=\mathrm{Ma}=(5 / 2) \mu \mathrm{Mg} \cos \theta$ giving $\mu=(2 / 7) \tan \theta$
b. Energy at the bottom is the same in both cases, however with $\mu=0$, there is no torque and no energy in rotation, which leaves more (all) energy in translation and velocity is higher

1977M2
a.

b. $\quad \Sigma \mathrm{F}_{\mathrm{y}}=0 ; \mathrm{T}+\mathrm{N}=\mathrm{W} ; \mathrm{N}=\mathrm{W}-\mathrm{T}=\mathrm{Mg}-(3 / 5) \mathrm{Mg}=(2 / 5) \mathrm{Mg}$
$\Sigma \mathrm{F}_{\mathrm{x}}=\mathrm{ma} ; \mathrm{F}_{\mathrm{f}}=\mathrm{ma} ; \mu \mathrm{N}=\mathrm{ma} ; 1 / 2(2 / 5) \mathrm{Mg}=\mathrm{Ma} ; \mathrm{a}=\mathrm{g} / 5$
c. $\quad \Sigma \tau=\mathrm{I} \alpha ;\left(\mathrm{T}-\mathrm{F}_{\mathrm{f}}\right) \mathrm{R}=1 / 2 \mathrm{MR}^{2} \alpha$
$(3 / 5) \mathrm{Mg}-(1 / 5) \mathrm{Mg}=1 / 2 \mathrm{MR} \alpha$
$(2 / 5) g=1 / 2 R \alpha$
$\alpha=4 \mathrm{~g} / 5 \mathrm{R}$
d. The cylinder is slipping on the surface and does not meet the condition for pure rolling

## 1980M3

a. $\quad \Sigma \mathrm{F}=\mathrm{ma} ; \mathrm{F}_{\mathrm{f}}=\mu \mathrm{F}_{\mathrm{N}}$;
$-\mu \mathrm{Mg}=\mathrm{Ma}$
$\mathrm{a}=-\mu \mathrm{g}$
$\mathrm{v}=\mathrm{v}_{0}+\mathrm{at}$
$\mathrm{v}=\mathrm{v}_{0}-\mu \mathrm{gt}$
b. $\tau=\mathrm{I} \alpha$ where the torque is provided by friction $\mathrm{F}_{\mathrm{f}}=\mu \mathrm{Mg}$
$\mu \mathrm{MgR}=\left(2 \mathrm{MR}^{2} / 5\right) \alpha$
$\alpha=(5 \mu \mathrm{~g} / 2 \mathrm{R})$
$\omega=\omega_{0}+\alpha \mathrm{t}=(5 \mu \mathrm{~g} / 2 \mathrm{R}) \mathrm{t}$
c. Slipping stops when the tangential velocity si equal to the velocity of the center of mass, or the condition for pure rolling has been met: $\mathrm{v}(\mathrm{t})=\omega(\mathrm{t}) \mathrm{R}$
$\mathrm{v}_{0}-\mu \mathrm{gt}=\mathrm{R}(5 \mathrm{~g} / 2 \mathrm{R}) \mathrm{t}$, which gives $\mathrm{T}=(2 / 7)\left(\mathrm{v}_{0} / \mu \mathrm{g}\right)$
d. Since the line of action of the frictional force passes through $P$, the net torque about point $P$ is zero. Thus, the time rate of change of the angular momentum is zero and the angular momentum is constant.

## 1986M2

a. $\quad \mathrm{U}=\mathrm{K}$

Mgh $=1 / 2 M v^{2}+1 / 2 I \omega^{2}$ and $\omega=v / R$
$\operatorname{Mgh}=1 / 2 \mathrm{Mv}^{2}+1 / 2(2 / 5) \mathrm{MR}^{2}(\mathrm{v} / \mathrm{R})^{2}=1 / 2 \mathrm{Mv}^{2}+(1 / 5) \mathrm{Mv}^{2}=7 \mathrm{Mv}^{2} / 10$
$\mathrm{v}^{2}=10 \mathrm{gh} / 7$
i. $\mathrm{K}_{\text {trans }}=1 / 2 \mathrm{Mv}^{2}=(5 / 7) \mathrm{Mgh}$
ii. $\mathrm{K}_{\text {rot }}=1 / 2 \mathrm{I} \omega^{2}=(2 / 7) \mathrm{Mgh}\left(\right.$ or $\left.\mathrm{Mgh}-\mathrm{K}_{\text {trans }}\right)$
b. i. $\tau=\mathrm{F}_{\mathrm{f}} \mathrm{R}=\mathrm{I} \alpha=\mathrm{I}(\mathrm{a} / \mathrm{R})$
$\mathrm{F}_{\mathrm{f}} \mathrm{R}=(2 / 5) \mathrm{MR}^{2}(\mathrm{a} / \mathrm{R})$
$\mathrm{F}_{\mathrm{f}}=(2 / 5) \mathrm{Ma}$
$\Sigma \mathrm{F}=\mathrm{ma}$
$M g \sin \theta-F_{f}=M a$
$M g \sin \theta-(2 / 5) M a=M a$
$\mathrm{g} \sin \theta=(7 / 5) \mathrm{a}$
$\mathrm{a}=(5 / 7) \mathrm{g} \sin \theta$
ii. $\mathrm{F}_{\mathrm{f}}=(2 / 5) \mathrm{Ma}=(2 / 5) \mathrm{M}(5 / 7) \mathrm{g} \sin \theta=(2 / 7) \mathrm{Mg} \sin \theta$
c. $\quad \mathrm{K}_{\text {tot }}=\mathrm{Mgh}$
d. Greater, the moment of inertia of the hollow sphere is greater and will be moving slower at the bottom of the incline. Since the translational speed is less, the translational KE is taking a smaller share of the same total energy as the solid sphere.

## 1990M2

a. $\mathrm{K}=\mathrm{U}$
$1 / 2 \mathrm{mv}_{0}{ }^{2}=\mathrm{mgH} ; \mathrm{H}=\mathrm{v}_{0}{ }^{2} / 2 \mathrm{~g}$
b. $\quad \mathrm{K}+\mathrm{W}_{\mathrm{f}}=\mathrm{U}$ where $\mathrm{W}_{\mathrm{f}}=-\mathrm{F}_{\mathrm{f}} \mathrm{d}$ and $\mathrm{F}_{\mathrm{f}}=\mu \mathrm{mg} \cos \theta$ and $\mathrm{d}=\mathrm{h} / \sin \theta$
$1 / 2 \mathrm{mv}_{0}{ }^{2}-(\mu \mathrm{mg} \cos \theta)(\mathrm{h} / \sin \theta)=\mathrm{mgh}$
$1 / 2 \operatorname{mv}_{0}{ }^{2}=\operatorname{mgh}(\mu \cot \theta+1)$
$\mathrm{h}=\mathrm{v}_{0}{ }^{2} /(2 \mathrm{~g}(\mu \cot \theta+1))=\mathrm{H} /(\mu \cot \theta+1)$
c. $K_{\text {trans }}+K_{\text {rot }}=U$ where $K_{\text {rot }}=1 / 2 I \omega^{2}=1 / 2\left(\mathrm{mR}^{2}\right)(\mathrm{v} / \mathrm{R})^{2}=1 / 2 \mathrm{mv}_{0}{ }^{2}$
$1 / 2 \mathrm{mv}_{0}{ }^{2}+1 / 2 \mathrm{mv}_{0}{ }^{2}=\mathrm{mgh}^{\prime}$
$\mathrm{h}^{\prime}=\mathrm{v}_{0}{ }^{2} / \mathrm{g}=2 \mathrm{H}$
d. Rotational energy will not change therefore $1 / 2 \mathrm{mv}_{0}{ }^{2}=\mathrm{mgh}^{\prime \prime}$ and $\mathrm{h}^{\prime \prime}=\mathrm{v}_{0}{ }^{2} / 2 \mathrm{~g}=\mathrm{H}$

## 1994M2

a. $\quad \mathrm{K}_{\text {tot }}=\mathrm{K}_{\text {trans }}+\mathrm{K}_{\text {rot }}=1 / 2 \mathrm{Mv}^{2}+1 / 2 \mathrm{I} \omega^{2}$ and $\omega=\mathrm{v} / \mathrm{R}$
$\mathrm{K}_{\text {tot }}=1 / 2 \mathrm{Mv}^{2}+1 / 2(2 / 5) \mathrm{MR}^{2}(\mathrm{v} / \mathrm{R})^{2}=1 / 2 \mathrm{Mv}^{2}+(1 / 5) \mathrm{Mv}^{2}=7 \mathrm{Mv}^{2} / 10=1750 \mathrm{~J}$
b. i. $\mathrm{K}_{\text {total, bottom }}=\mathrm{K}_{\text {top }}+\mathrm{U}_{\text {top }}=7 \mathrm{Mv}_{\text {top }}{ }^{2} / 10+\mathrm{Mgh} ; \mathrm{v}_{\text {top }}=7.56 \mathrm{~m} / \mathrm{s}$
ii. It is directed parallel to the incline: $25^{\circ}$
c. $y=y_{0}+v_{o y} t+1 / 2 a_{y} t^{2}$
$0 \mathrm{~m}=3 \mathrm{~m}+(7.56 \mathrm{~m} / \mathrm{s})\left(\sin 25^{\circ}\right) \mathrm{t}+1 / 2\left(-9.8 \mathrm{~m} / \mathrm{s}^{2}\right) \mathrm{t}^{2}$ which gives $\mathrm{t}=1.16 \mathrm{~s}$ (positive root)
$x=v_{x} t=(7.56 \mathrm{~m} / \mathrm{s})\left(\cos 25^{\circ}\right)(1.16 \mathrm{~s})=7.93 \mathrm{~m}$
d. The speed would be less than in $b$.

The gain in potential energy is entirely at the expense of the translational kinetic energy as there is no torque to slow the rotation.

## 1997M3

a. $\quad \mathrm{U}=\mathrm{K}$
$\mathrm{MgH}=1 / 2 M v^{2}+1 / 2 I \omega^{2}$ and $\omega=\mathrm{v} / \mathrm{R}$
$\mathrm{MgH}=1 / 2 \mathrm{Mv}^{2}+1 / 2(1 / 2) \mathrm{MR}^{2}(\mathrm{v} / \mathrm{R})^{2}=1 / 2 \mathrm{Mv}^{2}+1 / 4 \mathrm{Mv}^{2}=3 \mathrm{Mv}^{2} / 4$
$\mathrm{v}=(4 \mathrm{gH} / 3)^{1 / 2}$
b.

c. For a change of pace, we can use kinematics:
$\mathrm{v}_{\mathrm{f}}^{2}=\mathrm{v}_{\mathrm{i}}^{2}+2 \mathrm{ad}$
$4 \mathrm{gH} / 3=0+2 \mathrm{a}(\mathrm{H} / \sin \theta)$
$\mathrm{a}=(2 / 3) \mathrm{g} \sin \theta$
d. $\quad \Sigma \mathrm{F}=\mathrm{Ma}$
$M g \sin \theta-F_{f}=M a=M(2 / 3) g \sin \theta$
$M g \sin \theta-\mu M g \cos \theta=(2 / 3) M g \sin \theta$
$\mu \cos \theta=(1 / 3) \sin \theta$
$\mu=(1 / 3) \tan \theta$
e. i. The translational speed is greater, less energy is transferred to the rotational motion so more goes into the translational motion. Additionally, with a smaller frictional force, the translational acceleration is greater. ii. Total kinetic energy is less. Energy is dissipated as heat due to friction.

## 2002M2

a. For each tire: $\mathrm{I}=1 / 2 \mathrm{ML}^{2}=1 / 2(\mathrm{~m} / 4) \mathrm{r}^{2}$
$\mathrm{I}_{\text {total }}=4 \times \mathrm{I}=1 / 2 \mathrm{mr}^{2}$
b. $\mathrm{U}=\mathrm{K}$; total mass $=2 \mathrm{~m}$
$2 \mathrm{mgh}=1 / 2(2 \mathrm{~m}) \mathrm{v}^{2}+1 / 2 I \omega^{2}$ and $\omega=\mathrm{v} / \mathrm{R}$
$2 \mathrm{mgh}=\mathrm{mv}^{2}+1 / 2(1 / 2) \mathrm{mr}^{2}(\mathrm{v} / \mathrm{r})^{2}=1 / 2 \mathrm{mv}^{2}+(1 / 4) \mathrm{mv}^{2}=5 \mathrm{mv}^{2} / 4$
$\mathrm{v}=(8 \mathrm{gh} / 5)^{1 / 2}$
c. $\quad U_{g}=U_{s}$
$2 \mathrm{mgh}=1 / 2 \mathrm{kx}_{\mathrm{m}}{ }^{2} ; \mathrm{x}_{\mathrm{m}}=2(\mathrm{mgh} / \mathrm{k})^{1 / 2}$
d. In an inelastic collision, energy is lost. With less energy after the collision, the spring is compressed less.

## 2006M3

a. $\quad \Sigma \tau=\mathrm{I} \alpha$
$\mathrm{F}_{\mathrm{f}} \mathrm{R}=\mathrm{I} \alpha=\mathrm{MR}^{2}(\mathrm{a} / \mathrm{R}) ; \mathrm{F}_{\mathrm{f}}=\mathrm{Ma}$
$\Sigma \mathrm{F}=\mathrm{ma}$
$M g \sin \theta-F_{f}=M a$
$M g \sin \theta-M a=M a$
$\mathrm{a}=1 / 2 \mathrm{~g} \sin \theta$
b. $v_{f}^{2}=2 a L=g L \sin \theta$
$\mathrm{v}_{\mathrm{f}}=(\mathrm{gL} \sin \theta)^{1 / 2}$
c. $\quad \mathrm{H}=1 / 2 \mathrm{gt}^{2} ; \mathrm{t}=(2 \mathrm{H} / \mathrm{g})^{1 / 2}$
$\mathrm{d}=\mathrm{v}_{\mathrm{x}} \mathrm{t}=(\mathrm{gL} \sin \theta)^{1 / 2}(2 \mathrm{H} / \mathrm{g})^{1 / 2}=(2 \mathrm{LH} \sin \theta)^{1 / 2}$
d. Greater. A disk will have smaller rotational inertia and will therefore have a greater rotational velocity. This will lead to a greater translational velocity, and a greater distance $x$.

2010M2
a.

b. Torque provided by friction; $\mathrm{F}_{\mathrm{f}}=\mu \mathrm{F}_{\mathrm{N}}=\mu \mathrm{Mg} \cos \theta$
$\tau=\mathrm{F}_{\mathrm{f}} \mathrm{R}=\mathrm{I} \alpha=(2 / 5) \mathrm{MR}^{2}(\mathrm{a} / \mathrm{R}) ; \mathrm{F}_{\mathrm{f}}=(2 / 5) \mathrm{Ma} ; \mathrm{Ma}=(5 / 2) \mathrm{F}_{\mathrm{f}}$
$\Sigma \mathrm{F}=\mathrm{Ma}$
$\mathrm{Mg} \sin \theta-\mathrm{F}_{\mathrm{f}}=(5 / 2) \mathrm{F}_{\mathrm{f}}$
$\mathrm{F}_{\mathrm{f}}=(2 / 7) \mathrm{Mg} \sin \theta=8.4 \mathrm{~N}$
c. $\quad \mathrm{Mgh}=1 / 2 \mathrm{Mv}^{2}+1 / 2 \mathrm{I} \omega^{2}$ and $\omega=\mathrm{v} / \mathrm{R}$
$M g h=1 / 2 \mathrm{Mv}^{2}+1 / 2(2 / 5) \mathrm{MR}^{2}(\mathrm{v} / \mathrm{R})^{2}=1 / 2 \mathrm{Mv}^{2}+(1 / 5) \mathrm{Mv}^{2}=7 \mathrm{Mv}^{2} / 10$
$\mathrm{v}^{2}=10 \mathrm{gh} / 7 ; \mathrm{v}=5.3 \mathrm{~m} / \mathrm{s}$
d. The horizontal speed of the wagon is due to the horizontal component of the ball in the collision:
$\mathrm{M}_{\mathrm{i}} \mathrm{v}_{\mathrm{ix}}=\mathrm{M}_{\mathrm{f}} \mathrm{v}_{\mathrm{fx}}$; where $\mathrm{M}_{\mathrm{f}}=\mathrm{M}_{\text {ball }}+\mathrm{M}_{\text {wagon }}=18 \mathrm{~kg}$
$(6 \mathrm{~kg})(5.3 \mathrm{~m} / \mathrm{s})\left(\cos 30^{\circ}\right)=(18 \mathrm{~kg}) \mathrm{v}_{\mathrm{f}}$
$\mathrm{v}_{\mathrm{f}}=1.5 \mathrm{~m} / \mathrm{s}$

## SECTION D - Angular Momentum

## 1975M2

a. $\quad L_{i}=L_{f}$
$\mathrm{m}_{0} \mathrm{v}_{0} \mathrm{R} \sin \theta=\mathrm{I} \omega$
$\omega=m_{0} v_{0} R \sin \theta / I ; I=\left(M+m_{0}\right) R^{2}$
$\omega=\mathrm{m}_{0} \mathrm{v}_{0} \sin \theta /\left(\mathrm{M}+\mathrm{m}_{0}\right) \mathrm{R}$
b. $\quad \mathrm{K}_{\mathrm{i}}=1 / 2 \mathrm{~m}_{0} \mathrm{v}_{0}{ }^{2}$
$K_{f}=1 / 2 I \omega^{2}=1 / 2\left(M+m_{0}\right) R^{2}\left(m_{0} v_{0} \sin \theta /\left(M+m_{0}\right) R\right)^{2}$
$\mathrm{K}_{\mathrm{f}} / \mathrm{K}_{\mathrm{i}}=\mathrm{m}_{0} \sin ^{2} \theta /\left(\mathrm{M}+\mathrm{m}_{0}\right)$

## 1978M2

a. $\quad \mathrm{L}_{\mathrm{i}}=\mathrm{L}_{\mathrm{f}}$
$\mathrm{I} \omega=\mathrm{mvr}$
$(1 / 3) M_{1} \ell^{2} \omega=M_{2} v \ell$
$\mathrm{v}=\mathrm{M}_{1} \ell \omega / 3 \mathrm{M}_{2}$
b. $\quad \mathrm{p}_{\text {system }}=\mathrm{p}_{\mathrm{cm} \text { of rod }}=\mathrm{M}_{1} \mathrm{v}_{\mathrm{cm}}=\mathrm{M}_{1} \omega(\boldsymbol{\ell} / 2)$
c. $\quad \mathrm{P}_{\mathrm{f}}=\mathrm{M}_{2} \mathrm{v}_{\mathrm{f}}=\mathrm{M}_{1} \omega \ell / 3 \mathrm{M}_{2}$
d. There is a net external force on the system from the axis at point $P$.
e. Since the net external force acts at point P (the pivot), the net torque about point P is zero, hence angular momentum is conserved.

## 1981M3

a. $\quad \mathrm{m}_{2} \mathrm{v}=\mathrm{m}_{2}(-\mathrm{v} / 2)+\mathrm{M}_{1} \mathrm{v}^{\prime}$
$\mathrm{v}^{\prime}=3 \mathrm{~m}_{2} \mathrm{v} / 2 \mathrm{M}_{1}$
b. $\quad L_{i}=L_{f}$
$\mathrm{m}_{2} \mathrm{v}(\mathrm{L} / 3)=\mathrm{m}_{2}(-\mathrm{v} / 2)(\mathrm{L} / 3)+(1 / 12) \mathrm{M}_{1} \mathrm{~L}^{2} \omega$
$\omega=6 \mathrm{~m}_{2} \mathrm{v} / \mathrm{M}_{1} \mathrm{~L}$
c. $\quad \Delta \mathrm{K}=\mathrm{K}_{\mathrm{f}}-\mathrm{K}_{\mathrm{i}}=1 / 2 \mathrm{~m}_{2}(-\mathrm{v} / 2)^{2}+1 / 2 \mathrm{M}_{1} \mathrm{v}^{\prime 2}+1 / 2 \mathrm{I} \omega^{2}-1 / 2 \mathrm{~m}_{2} \mathrm{v}^{2}$
$=-3 \mathrm{~m}_{2} \mathrm{v}^{2} / 8+21 \mathrm{~m}_{2}^{2} \mathrm{v}^{2} / 8 \mathrm{M}_{1}$

## 1982M3

a. $\quad \mathrm{L}=\mathrm{I} \omega$ where $\mathrm{I}=\Sigma \mathrm{mr}^{2}=(2 \mathrm{~m}) \ell^{2}+\mathrm{m}(2 \ell)^{2}=6 \mathrm{~m} \ell^{2}$
$\mathrm{L}=6 \mathrm{~m} \ell^{2} \omega$
b. $\quad F_{f}=\mu \mathrm{mg}$
$\Sigma \tau=-(\mu(2 \mathrm{~m}) \mathrm{g} \ell+\mu \mathrm{mg}(2 \ell))=-4 \mu \mathrm{mg} \ell$
c. $\quad \alpha=\tau / \mathrm{I}=-4 \mu \mathrm{mg} \ell / 6 \mathrm{~m}^{2}=-2 \mu \mathrm{~g} / 3 \ell$
$\omega=\omega_{0}+\alpha$ t; setting $\omega=0$ and solving for $T$ gives $T=3 \omega_{0} \ell / 2 \mu g$

## 1987M3

a. $\mathrm{K}=\mathrm{U}$
$1 / 2 \mathrm{I} \omega^{2}=\mathrm{mgh}_{\mathrm{cm}}$
$1 / 2\left(\mathrm{~m} \boldsymbol{\ell}^{2} / 3\right) \omega^{2}=\operatorname{mg}(\boldsymbol{\ell} / 2)$ which gives $\omega=5 \mathrm{rad} / \mathrm{s}$
b. $\quad \mathrm{K}_{\mathrm{i}}=\mathrm{K}_{\mathrm{f}}$
$1 / 2 m_{0} v_{0}{ }^{2}=1 / 2 m_{0} v^{2}+1 / 2 I \omega^{2}$
$\mathrm{v}=8 \mathrm{~m} / \mathrm{s}$
c. $\quad \mathrm{L}=\mathrm{mvr}=(1 \mathrm{~kg})(10 \mathrm{~m} / \mathrm{s})(1.2 \mathrm{~m})=12 \mathrm{~kg}-\mathrm{m}^{2} / \mathrm{s}$
d. $\quad L_{i}=L_{f}$
$12 \mathrm{~kg}-\mathrm{m}^{2} / \mathrm{s}=\mathrm{m}_{0}\left(\mathrm{v}_{\perp}\right) \ell+\mathrm{I} \omega=\mathrm{m}_{0}(\mathrm{v} \cos \theta) \ell+\mathrm{I} \omega$
$\theta=60^{\circ}$

## 1992M2

a. $\quad \Sigma \tau=(3 \mathrm{M}+\mathrm{M}) \mathrm{g} \ell-\mathrm{Mg} \ell=3 \mathrm{Mg} \ell$
b. $\mathrm{I}=\Sigma \mathrm{mr}^{2}=4 \mathrm{Ml}^{2}+\mathrm{M}^{2}=5 \mathrm{~m} \ell^{2}$
$\alpha=\tau / \mathrm{I}=3 \mathrm{Mg} \ell / 5 \mathrm{~m} \ell^{2}=3 \mathrm{~g} / 5 \ell$
c. $\Delta \mathrm{U}_{\text {bug }}+\Delta \mathrm{U}_{\text {left sphere }}+\Delta \mathrm{U}_{\text {right sphere }}=\Delta \mathrm{K}_{\text {rot }}$
since $\Delta \mathrm{U}_{\text {left sphere }}=-\Delta \mathrm{U}_{\text {right sphere }}$, we only need to consider $\Delta \mathrm{U}_{\text {bug }}$
$3 \mathrm{Mg} \boldsymbol{\ell}=1 / 2 \mathrm{I} \omega^{2}=1 / 2\left(5 \mathrm{M} \boldsymbol{\ell}^{2}\right) \omega^{2}$
$\omega=(6 \mathrm{~g} / 5 \ell)^{1 / 2}$
d. $\quad \mathrm{L}=\mathrm{I} \omega=5 \mathrm{M} \ell^{2}(6 \mathrm{~g} / 5 \ell)^{1 / 2}=\left(30 \mathrm{M}^{2} \mathrm{~g} \ell^{3}\right)^{1 / 2}$
e. Let T be the force we are looking for
$\Sigma \mathrm{F}=\mathrm{ma}_{\mathrm{c}}$
$\mathrm{T}-3 \mathrm{Mg}=\mathrm{M} \omega^{2} \ell$
$\mathrm{T}=3 \mathrm{Mg}+3 \mathrm{M}(6 \mathrm{~g} / 5 \ell) \ell=33 \mathrm{Mg} / 5$

## 1996M3

a.
$I=\int r^{2} d m$
$d m=\frac{M}{l} d r$
$I=\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{M}{l} r^{2} d r$
$I=\left.\frac{M}{l} \frac{r^{3}}{3}\right|_{-l / 2} ^{l / 2}=\frac{M l^{2}}{12}$
b. $\mathrm{I}=\Sigma \mathrm{I}=\mathrm{M}^{2} / 12+\mathrm{M}(\boldsymbol{\ell} / 2)^{2}=\mathrm{M}^{2} / 3$
c./d./e.
$\Sigma \mathrm{F}=\mathrm{ma}$
for cat: $\mathrm{Mg}-\mathrm{T}=\mathrm{Ma}$
$\Sigma \tau=\mathrm{I} \alpha$ where $\alpha=\mathrm{a} / \mathrm{r}=\mathrm{a} /(\boldsymbol{l} / 2)$
for hoop: $\mathrm{T} \ell / 2=\left(\mathrm{M} \boldsymbol{\ell}^{2} / 3\right)(\mathrm{a} /(\boldsymbol{\ell} / 2))$ which gives $\mathrm{a}=3 \mathrm{~T} / 4 \mathrm{M}$
substituting gives $\mathrm{Mg}-\mathrm{T}=3 \mathrm{~T} / 4$
$\mathrm{T}=4 \mathrm{Mg} / 7$
$\alpha=\mathrm{T} \ell / 2 \mathrm{I}=6 \mathrm{~g} / 7 \boldsymbol{\ell}$
$\mathrm{a}=\alpha(\boldsymbol{\ell} / 2)=3 \mathrm{~g} / 7$
f. $\quad \mathrm{L}=\operatorname{Mv}(\ell / 2)$ where v is found from $\mathrm{v}^{2}=\mathrm{v}_{0}{ }^{2}+2 \mathrm{aH}=2(3 \mathrm{~g} / 7)(5 \ell / 3)=10 \mathrm{~g} \ell / 7$
$\mathrm{L}=1 / 2 \mathrm{Ml}(10 \mathrm{~g} \ell / 7)^{1 / 2}$

## 1998M2

a. i. $\mathrm{mv}_{0}=(3 \mathrm{~m}) \mathrm{v}_{\mathrm{f}} ; \mathrm{v}_{\mathrm{f}}=\mathrm{V}_{0} / 3$
$\mathrm{K}_{\mathrm{f}}=1 / 2(3 \mathrm{~m})\left(\mathrm{v}_{0} / 3\right)^{2}=\mathrm{mv}_{0}{ }^{2} / 6$
ii. $\Delta \mathrm{K}=\mathrm{K}_{\mathrm{f}}-\mathrm{K}_{\mathrm{i}}=\mathrm{mv}_{0}{ }^{2} / 6-1 / 2 \mathrm{mv}_{0}^{2}=-\mathrm{mv}_{0}^{2} / 3$
b. i. $\mathrm{r}_{\mathrm{cm}}=\Sigma \mathrm{m}_{\mathrm{i}} \mathrm{r}_{\mathrm{i}} / \Sigma \mathrm{m}=\mathrm{m}(0)+2 \mathrm{~m}(\boldsymbol{\ell}) /(\mathrm{m}+2 \mathrm{~m})=(2 / 3) \ell$
ii.

iii. $\mathrm{p}_{\mathrm{i}}=\mathrm{p}_{\mathrm{f}}$
$\mathrm{mv}_{0}=(3 \mathrm{~m}) \mathrm{v}_{\mathrm{f}} ; \mathrm{v}_{\mathrm{f}}=\mathrm{v}_{0} / 3$
iv. $L_{i}=L_{f}$
$\mathrm{mv}_{0} \mathrm{R} \sin \theta=\mathrm{mv}_{0}(\boldsymbol{\ell} / 3)=\mathrm{I} \omega$ where $\mathrm{I}=\Sigma \mathrm{mr}^{2}=\mathrm{m}(2 \ell / 3)^{2}+2 \mathrm{~m}(\ell / 3)^{2}=(2 / 3) \mathrm{m}^{2}$
solving yields $\omega=\mathrm{V}_{0} / 2 \ell$
v. $\mathrm{K}_{\mathrm{i}}=1 / 2 \mathrm{mv}_{0}{ }^{2}$
$\mathrm{K}_{\mathrm{f}}=1 / 2 \mathrm{mv}_{\mathrm{f}}^{2}+1 / 2 \mathrm{I} \omega^{2}=1 / 2(3 \mathrm{~m})\left(\mathrm{v}_{0} / 3\right)^{2}+1 / 2(2 / 3) \mathrm{m}^{2}\left(\mathrm{v}_{0} / 2 \ell\right)^{2}=1 / 4 \mathrm{mv}_{0}{ }^{2}$
$\Delta \mathrm{K}=-1 / 4 \mathrm{mv}_{0}{ }^{2}$

2005M3
a. $\quad \mathrm{L}=\mathrm{I} \omega=(1 / 3) \mathrm{M}_{1} \mathrm{~d}^{2} \omega$
b. $\quad L_{f}=L_{i}$
$\mathrm{M}_{2} \mathrm{vd}=(1 / 3) \mathrm{M}_{1} \mathrm{~d}^{2} \omega$
$\mathrm{v}=\mathrm{M}_{1} \mathrm{~d} \omega / 3 \mathrm{M}_{2}$
c. $\quad K_{f}=K_{i}$
$1 / 2 M_{2} v^{2}=1 / 2 I \omega^{2}$
$\mathrm{M}_{2} \mathrm{v}^{2}=\mathrm{I} \omega^{2}$
$M_{2}\left(M_{1} d \omega / 3 M_{2}\right)=(1 / 3) M_{1} d^{2} \omega^{2}$
$M_{2}(1 / 9)\left(M_{1} / M_{2}\right)^{2} d^{2} \omega^{2}=(1 / 3) M_{1} d^{2} \omega^{2}$
$(1 / 9)\left(M_{1}{ }^{2} / M_{2}\right)=M_{1} / 3$
$\mathrm{M}_{1} / \mathrm{M}_{2}=3$
d. $\quad L_{f}=L_{i}$
$\mathrm{M}_{1} \mathrm{vx}=(1 / 3) \mathrm{M}_{1} \mathrm{~d}^{2} \omega$
$\mathrm{v}=\mathrm{d}^{2} \omega / 3 \mathrm{x}$
$1 / 2 M_{1} v^{2}=1 / 2 I \omega^{2}$
$M_{1} v^{2}=I \omega^{2}=(1 / 3) M_{1} d^{2} \omega^{2}$
$v^{2}=d^{2} \omega^{2} / 3$
substituting from above $\quad\left(d^{2} \omega / 3 x\right)^{2}=d^{2} \omega^{2} / 3$
solving for x gives $\mathrm{x}=\mathrm{d} / \sqrt{ } 3$

